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ORDINARY QUADRATIC DIFFERENTIAL
EQUATIONS WITH MAXIMA

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ABSTRACT. In this paper the authors prove the existence as well as approximation of the positive solutions for an initial value problem of first order ordinary nonlinear quadratic differential equations with maxima. An algorithm for the solutions is developed and it is shown that the sequence of successive approximations converges monotonically to the positive solution of the related quadratic differential equations under some suitable mixed hybrid conditions. We base our results on the Dhage iteration method embodied in a recent hybrid fixed point theorem of Dhage (2014) in a partially ordered normed linear space. An example is also provided to illustrate the hypotheses and abstract theory developed in this paper.

1. Introduction

Given a closed and bounded interval \(J = [t_0, t_0 + a]\), of the real line \(\mathbb{R}\) for some \(t_0, a \in \mathbb{R}\) with \(t_0 \geq 0, a > 0\), consider the initial value problem (in short IVP) of first order ordinary nonlinear quadratic differential equation (in short QDE) with maxima viz.,

\[
\frac{d}{dt} \left[ \frac{x(t)}{f(t, x(t), X(t))} \right] + \lambda \left[ \frac{x(t)}{f(t, x(t), X(t))} \right] = g(t, x(t), X(t)), \quad t \in J, \\
x(t_0) = x_0 \in \mathbb{R}_+,
\]

\(1.1\)
for \( \lambda \in \mathbb{R}, \lambda > 0 \), where \( f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{ 0 \} \) and \( g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are continuous functions and \( X(t) = \max_{t_0 \leq \xi \leq t} x(\xi) \) for \( t \in J \).

By a solution of the QDE (1.1) we mean a function \( x \in C^1(J, \mathbb{R}) \) that satisfies

(i) \( t \mapsto \frac{x}{f(t, x, y)} \) is a continuously differentiable function for each \( x, y \in \mathbb{R} \), and

(ii) \( x \) satisfies the equations in (1.1) on \( J \),

where \( C^1(J, \mathbb{R}) \) is the space of continuously differentiable real-valued functions defined on \( J \).

The origin of the differential equations with maxima lies in the real world problems of automatic regulation of the technical systems and such differential equations is a special class of functional differential equations in which the present state of the unknown function related to the systems depends upon the maximum value of the past state in some past interval of time. See Magomedov [16, 17] for the details. Again, Mishkis [18] pointed out in his survey the necessity of the study of differential equations with maxima and since then several classes of ordinary and partial differential equations with maxima have been discussed in the literature for different qualitative aspects of the solutions. See Bainov and Hristova [1], Otrocol and Rus [14], Dhage and Otrocol [13] and the references therein. Similarly, the study of quadratic differential equations which is a special class of hybrid differential equations may be found in the works of Dhage [2, 4] wherein such quadratic differential equations are discussed for different aspects of the solutions. See Dhage and Lakshmikantham [12], Dhage and Dhage [11] and the references therein. The purpose of the present paper is to blend these two ideas together and discuss the quadratic differential equations with maxima for existence and numerical aspects of the solutions. It is well-known that the hybrid differential equations can be tackled using the Dhage iteration method embodied in the hybrid fixed point theory initiated by Dhage [2, 5, 6]) which also yields the algorithms for the solutions. Therefore, it is of interest to establish algorithms for the quadratic differential equations with maxima (1.1) for existence and approximation of the solutions along similar lines. The novelty of the present paper lies in the fact that our problem (1.1) as well as our method is new to the literature in the theory of nonlinear differential equations with maxima.

2. Auxiliary results

In this section we give all the preliminaries and key tool that is used in subsequent part of the paper. Unless otherwise mentioned, throughout this paper that follows, let \( E \) denote a partially ordered real normed linear space with an order relation \( \preceq \) and the norm \( \| \cdot \| \). It is known that \( E \) is regular if \( \{ x_n \}_{n \in \mathbb{N}} \) is a nondecreasing (resp. nonincreasing) sequence in \( E \) such that \( x_n \to x^* \) as \( n \to \infty \), then \( x_n \preceq x^* \) (resp. \( x_n \succeq x^* \)) for all
Clearly, the partially ordered Banach space $C(J, \mathbb{R})$ is regular and the conditions guaranteeing the regularity of any partially ordered normed linear space $E$ may be found in Heikkilä and Lakshmikantham [15] and the references therein.

We need the following definitions in the sequel.

**Definition 2.1.** A mapping $T : E \to E$ is called isotone or nondecreasing if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in E$.

**Definition 2.2** (Dhage [6]). A mapping $T : E \to E$ is called partially continuous at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|Tx - Ta\| < \epsilon$ whenever $x$ is comparable to $a$ and $\|x - a\| < \delta$. $T$ is called partially continuous on $E$ if it is partially continuous at every point of it. It is clear that if $T$ is partially continuous on $E$, then it is continuous on every chain $C$ contained in $E$.

**Definition 2.3** (Dhage [6]). A non-empty subset $S$ of the partially ordered Banach space $E$ is called partially bounded if every chain $C$ in $S$ is bounded. An operator $T : E \to E$ is called partially bounded if every chain $C$ in $T(E)$ is bounded. $T$ is called uniformly partially bounded if all chains $C$ in $T(E)$ are bounded by a unique constant. $T$ is called bounded if $T(E)$ is a bounded subset of $E$.

**Definition 2.4** (Dhage [6]). A non-empty subset $S$ of the partially ordered Banach space $E$ is called partially compact if every chain or totally ordered set $C$ in $T(E)$ is a relatively compact subset of $E$. $T$ is called uniformly partially compact if $T(E)$ is a uniformly partially bounded and partially compact on $E$. $T$ is called partially totally bounded if for any bounded subset $S$ of $E$, $T(S)$ is a relatively compact subset of $E$. If $T$ is partially continuous and partially totally bounded, then it is called partially completely continuous on $E$.

**Remark 2.1.** Suppose that $T$ is a nondecreasing operator on $E$ into itself. Then $T$ is a partially bounded or partially compact if $T(C)$ is a bounded or relatively compact subset of $E$ for each chain $C$ in $E$.

**Definition 2.5** (Dhage [3]). The order relation $\preceq$ and the metric $d$ on a non-empty set $E$ are said to be compatible if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in $E$ and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $x^*$ implies that the original sequence $\{x_n\}$ converges to $x^*$. Similarly, given a partially ordered normed linear space $(E, \preceq, \| \cdot \|)$, the order relation $\preceq$ and the norm $\| \cdot \|$ are said to be compatible if $\preceq$ and the metric $d$ defined through the norm $\| \cdot \|$ are compatible. A subset $S$ of $E$ is called Janhavi if the order relation $\preceq$ and the metric $d$ or the norm $\| \cdot \|$ are compatible in it. In particular, if $S = E$, then $E$ is called a Janhavi metric or Janhavi Banach space.
Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a Janhavi Banach space.

**Definition 2.6** (Dhage [6]). An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $T : E \to E$ is called **partially nonlinear $\mathcal{D}$-Lipschitz** if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that
\[
\|Tx - Ty\| \leq \psi(\|x - y\|) \tag{2.1}
\]
for all comparable elements $x, y \in E$. If $\psi(r) = kr$, $k > 0$, then $T$ is called a partially Lipschitz with a Lipschitz constant $k$.

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote
\[
E^+ = \{ x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E \}
\]
and
\[
\mathcal{K} = \{ E^+ \subset E \mid uv \in E^+ \text{ for all } u, v \in E^+ \}. \tag{2.2}
\]
The elements of the set $\mathcal{K}$ are called the positive vectors in $E$. The following lemma follows immediately from the definition of the set $\mathcal{K}$ which is often times used in the hybrid fixed point theory of Banach algebras and applications to nonlinear quadratic differential and integral equations.

**Lemma 2.1** (Dhage [3]). If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.

**Definition 2.7.** An operator $T : E \to E$ is said to be positive if the range $R(T)$ of $T$ is such that $R(T) \subseteq \mathcal{K}$.

The Dhage iteration principle or method (in short DIP or DIM) developed in Dhage [5, 6] may be described as “the monotonic convergence of the sequence of successive approximations to the solutions of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial or first approximation” and which forms a useful tool in the subject of existence theory of nonlinear analysis. It is known that the Dhage iteration method is different from other iterations methods and embodied in the following applicable hybrid fixed point theorem of Dhage [6] which is the key tool for our work contained in the present paper. A few other hybrid fixed point theorems containing the Dhage iteration method along with applications appear in Dhage [5, 6].

**Theorem 2.1** (Dhage [5, 6]). Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear algebra such that every compact chain of $E$ is Janhavi. Let $A, B : E \to \mathcal{K}$ be two nondecreasing operators such that

(a) $A$ is partially bounded and partially nonlinear $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function $\psi_A$, 

...
(b) \( \mathcal{B} \) is partially continuous and uniformly partially compact,
(c) \( 0 < M\psi_A(r) < r, r > 0, \) where \( M = \sup\{\|\mathcal{B}(C)\| : C \) is a chain in \( E \}, \)
and
(d) there exists an element \( x_0 \in X \) such that \( x_0 \preceq A x_0 \mathcal{B} x_0 \) or \( x_0 \succeq A x_0 \mathcal{B} x_0. \)

Then the operator equation
\[
A x \mathcal{B} x = x
\]
has a positive solution \( x^* \) in \( E \) and the sequence \( \{x_n\} \) of successive iterations defined by \( x_{n+1} = A x_n \mathcal{B} x_n, \ n = 0, 1, \ldots; \) converges monotonically to \( x^*. \)

**Remark 2.2.** The condition that every compact chain of \( E \) is Janhavi holds if every partially compact subset of \( E \) possesses the compatibility property with respect to the order relation \( \preceq \) and the norm \( \| \cdot \| \) in it.

**Remark 2.3.** We remark that hypothesis (a) of Theorem 2.1 implies that the operator \( A \) is partially continuous and consequently both the operators \( A \) and \( \mathcal{B} \) in the theorem are partially continuous on \( E. \) The regularity of \( E \)
in above Theorem 2.1 may be replaced with a stronger continuity condition of the operators \( A \) and \( \mathcal{B} \) on \( E \) which is a result proved in Dhage [3, 4].

In the following section we give the main existence approximation result of this paper.

### 3. Main results

The QDE (1.1) is considered in the function space \( C(\mathcal{J}, \mathbb{R}) \) of continuous real-valued functions defined on \( \mathcal{J}. \) We define a norm \( \| \cdot \| \) and the order relation \( \preceq \) in \( C(\mathcal{J}, \mathbb{R}) \) by
\[
\|x\| = \sup_{t \in \mathcal{J}} |x(t)| \quad (3.1)
\]
and
\[
x \preceq y \iff x(t) \leq y(t) \quad (3.2)
\]
for all \( t \in \mathcal{J} \) respectively. Clearly, \( C(\mathcal{J}, \mathbb{R}) \) is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation \( \preceq. \) It is known that the partially ordered Banach algebra \( C(\mathcal{J}, \mathbb{R}) \) has some nice properties w.r.t. the above order relation in it. The following lemma follows by an application of Arzelá-Ascoli theorem.

**Lemma 3.1.** Let \( (C(\mathcal{J}, \mathbb{R}), \preceq, \| \cdot \|) \) be a partially ordered Banach space with the norm \( \| \cdot \| \) and the order relation \( \preceq \) defined by (3.1) and (3.2) respectively. Then every partially compact subset \( S \) of \( C(\mathcal{J}, \mathbb{R}) \) is Janhavi.

**Proof.** The proof of the lemma is given in Dhage and Dhage [7, 8, 9, 10] and so we omit the details of it.

We need the following definition in what follows.
**Definition 3.1.** A function \( u \in C^1(J, \mathbb{R}) \) is said to be a lower solution of the QDE (1.1) if the function \( t \mapsto \frac{u(t)}{f(t, u(t), U(t))} \) is continuously differentiable for each \( x, y \in \mathbb{R} \) and satisfies

\[
\frac{d}{dt} \left( \frac{u(t)}{f(t, u(t), U(t))} \right) + \lambda \left( \frac{u(t)}{f(t, u(t), U(t))} \right) \leq g \left( t, u(t), U(t) \right),
\]

for all \( t \in J \), where \( U(t) = \max_{t_0 \leq \xi \leq t} u(\xi) \) for \( t \in J \). Similarly, a function \( v \in C^1(J, \mathbb{R}) \) is said to be an upper solution of the QDE (1.1) if it satisfies the above property and inequalities with reverse sign.

We consider the following set of assumptions in what follows:

(A0) The map \( x \mapsto \frac{x}{f(t, x, x)} \) is injection for each \( t \in J \).

(A1) \( f \) defines a function \( f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ \).

(A2) There exists a constant \( M_f > 0 \) such that \( 0 < f(t, x, y) \leq M_f \) for all \( t \in J \) and \( x, y \in \mathbb{R} \).

(A3) There exists a \( \mathcal{D} \)-function \( \varphi \) such that

\[
0 \leq f(t, x_1, x_2) - f(t, y_1, y_2) \leq \varphi \left( \max \{x_1 - y_1, x_2 - y_2\} \right),
\]

for all \( t \in J \) and \( x_1, x_2, y_1, y_2 \in \mathbb{R} \), \( x_1 \geq y_1 \) and \( x_2 \geq y_2 \).

(B1) \( g \) defines a function \( g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ \).

(B2) There exists a constant \( M_g > 0 \) such that \( g(t, x, y) \leq M_g \) for all \( t \in J \) and \( x, y \in \mathbb{R} \).

(B3) \( g(t, x, y) \) is nondecreasing in \( x \) and \( y \) for all \( t \in J \).

(B4) The QDE (1.1) has a lower solution \( u \in C^1(J, \mathbb{R}) \).

**Remark 3.1.** Notice that Hypothesis (A0) holds in particular if the function \( x \mapsto \frac{x}{f(t, x, x)} \) is increasing for each \( t \in J \).

**Lemma 3.2.** Suppose that hypothesis (A0) holds. Then a function \( x \in C(J, \mathbb{R}) \) is a solution of the QDE (1.1) if and only if it is a solution of the nonlinear quadratic integral equation (in short QIE),

\[
x(t) = \left[ f \left( t, x(t), X(t) \right) \right] \left( c e^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x(s), X(s)) \, ds \right)
\]

for all \( t \in J \), where \( c = \frac{x_0 e^{\lambda t_0}}{f(t_0, x_0, x_0)} \).

**Theorem 3.1.** Assume that hypotheses (A0)-(A3) and (B1)-(B4) hold. Furthermore, assume that

\[
\left( \left| \frac{x_0}{f(t_0, x_0, x_0)} \right| + M_g a \right) \varphi(r) < r, \quad r > 0,
\]

(3.4)
then the QDE (1.1) has a positive solution \( x^* \) defined on \( J \) and the sequence \( \{x_n\}_{n=1}^\infty \) of successive approximations defined by

\[
x_{n+1}(t) = \left[ f(t, x_n(t), X_n(t)) \right] \left( ce^{-\lambda t} + \int_{t_0}^t e^{-\lambda(t-s)} g(s, x_n(s), X_n(s)) \, ds \right),
\]

(3.5)

where \( x_1 = u \), converges monotonically to \( x^* \).

**Proof.** Set \( E = C(J, \mathbb{R}) \) Then, by Lemma 3.1, every compact chain in \( E \) possesses the compatibility property with respect to the norm \( \| \cdot \| \) and the order relation \( \leq \) in \( E \).

Define two operators \( A \) and \( B \) on \( E \) by

\[
A x(t) = f(t, x(t), X(t)), \quad t \in J,
\]

(3.6)

and

\[
B x(t) = ce^{-\lambda t} + \int_{t_0}^t e^{-\lambda(t-s)} g(s, x(s), X(s)) \, ds, \quad t \in J.
\]

(3.7)

From the continuity of the integral, it follows that \( A \) and \( B \) define the maps \( A, B : E \to K \). Now by Lemma 3.2, the QDE (1.1) is equivalent to the operator equation

\[
A x(t) B x(t) = x(t), \quad t \in J.
\]

(3.8)

We shall show that the operators \( A \) and \( B \) satisfy all the conditions of Theorem 2.1. This is achieved in the series of following steps.

**Step I:** \( A \) and \( B \) are nondecreasing on \( E \).

Let \( x, y \in E \) be such that \( x \geq y \). Then \( x(t) \geq y(t) \) for all \( t \in J \). Since \( y \) is continuous on \([a, t]\), there exists a \( \xi^* \in [a, t] \) such that \( y(\xi^*) = \max_{a \leq \xi \leq t} y(\xi) \).

By definition of \( \leq \), one has \( x(\xi^*) \geq y(\xi^*) \). Consequently, we obtain

\[
X(t) = \max_{t_0 \leq \xi \leq t} x(\xi) \geq x(\xi^*) \geq y(\xi^*) = \max_{t_0 \leq \xi \leq t} y(\xi) = Y(t)
\]

for each \( t \in J \). Then by hypothesis (A3), we obtain

\[
A x(t) = f(t, x(t), X(t)) \geq f(t, x(t), Y(t)) = A y(t),
\]

for all \( t \in J \). This shows that \( A \) is nondecreasing operator on \( E \) into \( E \). Similarly using hypothesis (B3), it is shown that the operator \( B \) is also nondecreasing on \( E \) into itself. Thus, \( A \) and \( B \) are nondecreasing positive operators on \( E \) into itself.

**Step II:** \( A \) is partially bounded and partially \( D \)-Lipschitz on \( E \).

Let \( x \in E \) be arbitrary. Then by (A2),

\[
|A x(t)| \leq |f(t, x(t), X(t))| \leq M_f,
\]

for all \( t \in J \). Taking supremum over \( t \), we obtain \( \|A x\| \leq M_f \) and so, \( A \) is bounded. This further implies that \( A \) is partially bounded on \( E \).
Next, let \( x, y \in E \) be such that \( x \geq y \). Then, we have \( |x(t) - y(t)| \leq |X(t) - Y(t)| \) and that
\[
|X(t) - Y(t)| = |x(t) - y(t)| \\
= \max_{t_0 \leq \xi \leq t} x(\xi) - \max_{t_0 \leq \xi \leq t} y(\xi) \\
\leq \max_{t_0 \leq \xi \leq t} (x(\xi) - y(\xi)) \\
= \max_{t_0 \leq \xi \leq t} |x(\xi) - y(\xi)| \\
\leq \|x - y\|
\]
for each \( t \in J \). As a result, we obtain
\[
|Ax(t) - Ay(t)| = |f(t, x(t), X(t)) - f(t, y(t), Y(t))| \\
\leq \varphi(\max\{|x(t) - y(t)|, |X(t) - Y(t)|\}) \\
\leq \varphi(\|x - y\|),
\]
for all \( t \in J \). Taking supremum over \( t \), we obtain
\[
\|Ax - Ay\| \leq \varphi(\|x - y\|),
\]
for all \( x, y \in E \) with \( x \geq y \). Hence, \( A \) is a partial nonlinear \( \mathcal{D} \)-Lipschitz on \( E \) which further also implies that \( A \) is a partially continuous on operator on \( E \).

**Step III:** \( \mathcal{B} \) is partially continuous on \( E \).

Let \( \{x_n\}_{n \in \mathbb{N}} \) be a sequence in a chain \( C \) of \( E \) such that \( x_n \to x \) for all \( n \in \mathbb{N} \). Then, by dominated convergence theorem, we have
\[
\lim_{n \to \infty} \mathcal{B}x_n(t) = \lim_{n \to \infty} ce^{-\lambda t} + \lim_{n \to \infty} \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x_n(s), X_n(s)) \, ds \\
= ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} \left[ \lim_{n \to \infty} g(s, x_n(s), X_n(s)) \right] \, ds \\
= ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x(s), X(s)) \, ds \\
= \mathcal{B}x(t),
\]
for all \( t \in J \). This shows that \( \mathcal{B}x_n \) converges to \( \mathcal{B}x \) pointwise on \( J \).

Next, we will show that \( \{\mathcal{B}x_n\}_{n \in \mathbb{N}} \) is an equicontinuous sequence of functions in \( E \). Let \( t_1, t_2 \in J \) with \( t_1 < t_2 \). Then
\[
|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \leq \left| ce^{-\lambda t_1} - ce^{-\lambda t_2} \right| \\
+ \left| \int_{t_0}^{t_1} e^{-\lambda(t_1-s)} g(s, x_n(s), X_n(s)) \, ds - \int_{t_0}^{t_1} e^{-\lambda(t_2-s)} g(s, x_n(s), X_n(s)) \, ds \right| \\
+ \left| \int_{t_0}^{t_1} e^{-\lambda(t_2-s)} g(s, x_n(s), X_n(s)) \, ds - \int_{t_0}^{t_2} e^{-\lambda(t_2-s)} g(s, x_n(s), X_n(s)) \, ds \right|
\]
\begin{align*}
\leq & \left| ce^{-\lambda t_1} - ce^{-\lambda t_2} \right| + \left| \int_{t_0}^{t_1} e^{-\lambda(t_1-s)} - e^{-\lambda(t_2-s)} \left| g(s, x_n(s), X_n(s)) \right| ds \right| \\
& + \left| \int_{t_0}^{t_2} g(s, x_n(s), X_n(s)) \right| ds \\
\leq & \left| ce^{-\lambda t_1} - ce^{-\lambda t_2} \right| + M_g \int_{t_0}^{t_0+a} e^{-\lambda(t_1-s)} - e^{-\lambda(t_2-s)} \right| ds \\
& + M_g \left| t_1 - t_2 \right| \\
\rightarrow 0 & \text{ as } t_2 - t_1 \rightarrow 0
\end{align*}

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $Bx_n \rightarrow Bx$ is uniform and hence $B$ is partially continuous on $E$.

**Step IV:** $B$ is uniformly partially compact operator on $E$.

Let $C$ be an arbitrary chain in $E$. We show that $B(C)$ is a uniformly bounded and equicontinuous set in $E$. First we show that $B(C)$ is uniformly bounded. Let $y \in B(C)$ be any element. Then there is an element $x \in C$ such that $y = Bx$. Now, by hypothesis (B2),

\begin{align*}
\left| y(t) \right| & \leq \left| ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x(s), X(s)) ds \right| \\
& \leq \left| ce^{-\lambda t} \right| + \left| \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x(s), X(s)) ds \right| \\
& \leq \frac{x_0}{J(t_0, x_0, x_0)} + \left| \int_{t_0}^{t_0+a} g(s, x(s), X(s)) ds \right| \\
& \leq \frac{x_0}{f(t_0, x_0, x_0)} + M_g a = M,
\end{align*}

for all $t \in J$. Taking supremum over $t$, we obtain $\|y\| = \|Bx\| \leq M$ for all $y \in B(C)$. Hence, $B(C)$ is a uniformly bounded subset of $E$. Moreover, $\|B(C)\| \leq M$ for all chains $C$ in $E$. Hence, $B$ is a uniformly partially bounded operator on $E$.

Next, we will show that $B(C)$ is an equicontinuous set in $E$. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for any $y \in B(C)$, one has

\begin{align*}
\left| y(t_2) - y(t_1) \right| & = \left| Bx(t_2) - Bx(t_1) \right| \\
& \leq \left| ce^{-\lambda t_1} - ce^{-\lambda t_2} \right| \\
& + \left| \int_{t_0}^{t_1} e^{-\lambda(t_1-s)} g(s, x(s), X(s)) ds \right| - \left| \int_{t_0}^{t_1} e^{-\lambda(t_2-s)} g(s, x(s), X(s)) ds \right| \\
& + \left| \int_{t_0}^{t_1} e^{-\lambda(t_2-s)} g(s, x(s), X(s)) ds \right| - \left| \int_{t_0}^{t_2} e^{-\lambda(t_2-s)} g(s, x(s), X(s)) ds \right| \\
& \leq \left| ce^{-\lambda t_1} - ce^{-\lambda t_2} \right| + \left| \int_{t_0}^{t_1} e^{-\lambda(t_1-s)} - e^{-\lambda(t_2-s)} \left| g(s, x(s), X(s)) \right| ds \right|
\end{align*}
uniformly for all \( y \in \mathcal{B}(C) \). Hence \( \mathcal{B}(C) \) is an equicontinuous subset of \( E \). Now, \( \mathcal{B}(C) \) is a uniformly bounded and equicontinuous set of functions in \( E \), so it is compact. Consequently, \( \mathcal{B} \) is a uniformly partially compact operator on \( E \) into itself.

**Step V:** \( u \) satisfies the operator inequality \( u \leq AuBu \).

By hypothesis (B4), the QDE (1.1) has a lower solution \( u \) defined on \( J \). Then, we have

\[
\frac{d}{dt} \left[ \frac{u(t)}{f(t, u(t), U(t))} \right] + \lambda \left[ \frac{u(t)}{f(t, u(t), U(t))} \right] \leq g(t, u(t), U(t)),
\]

\[\forall t \in J,\ u(t_0) \leq x_0,\ (3.9)\]

for all \( t \in J \). Multiplying the above inequality (3.7) by the integrating factor \( e^{\lambda t} \), we obtain

\[
\left( e^{\lambda t} \frac{u(t)}{f(t, u(t), U(t))} \right)' \leq e^{\lambda t} g(t, u(t), U(t)), \tag{3.10}
\]

for all \( t \in J \). A direct integration of (3.8) from \( t_0 \) to \( t \) yields

\[
u(t) \leq \left[ f(t, u(t), U(t)) \right] \left( ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, u(s), U(s)) \, ds \right), \tag{3.11}\]

for all \( t \in J \). From definitions of the operators \( \mathcal{A} \) and \( \mathcal{B} \) it follows that \( u(t) \leq \mathcal{A}u(t)Bu(t), \) for all \( t \in J \). Hence \( u \leq \mathcal{A}uBu \).

**Step VI:** \( \mathcal{D} \)-function \( \varphi \) satisfies the growth condition \( 0 < M\psi(\mathcal{A})(r) < r, \) \( \forall r > 0. \)

Finally, the \( \mathcal{D} \)-function \( \varphi \) of the operator \( \mathcal{A} \) satisfies the inequality given in hypothesis (d) of Theorem 2.1. Now from the estimate given in Step IV, it follows that

\[
M\psi(\mathcal{A})(r) \leq \left( \frac{x_0}{f(t_0, x_0, x_0)} + M_g a \right) \varphi(r) < r
\]

for all \( r > 0. \)

Thus \( \mathcal{A} \) and \( \mathcal{B} \) satisfy all the conditions of Theorem 2.1 and we apply it to conclude that the operator equation \( \mathcal{A}x \mathcal{B}x = x \) has a solution. Consequently the integral equation and the QDE (1.1) has a positive solution \( x^* \)
defined on \( J \). Furthermore, the sequence \( \{x_n\}_{n=1}^{\infty} \) of successive approximations defined by (3.5) converges monotonically to \( x^* \). This completes the proof.

**Remark 3.2.** The conclusion of Theorem 3.1 also remains true if we replace the hypothesis with the following:

\((B'_4)\) The QDE (1.1) has an upper solution \( v \in C^1(J, \mathbb{R}) \).

The proof under the new hypothesis is similar to the proof of Theorem 3.1 with appropriate modifications.

**Remark 3.3.** We note that if the QDE (1.1) has a lower solution \( u \) as well as an upper solution \( v \) such that \( u \leq v \), then under the given conditions of Theorem 3.1 it has corresponding solutions \( x_* \) and \( x^* \) and these solutions satisfy \( x_* \leq x^* \). Hence they are the minimal and maximal solutions of the PBVP (1.1) in the vector segment \([u, v]\) of the Banach space \( E = C^1(J, \mathbb{R}) \), where the vector segment \([u, v]\) is a set in \( C^1(J, \mathbb{R}) \) defined by

\[ [u, v] = \{ x \in C^1(J, \mathbb{R}) \mid u \leq x \leq v \}. \]

This is because the order relation \( \leq \) defined by (3.2) is equivalent to the order relation defined by the order cone \( \mathcal{K} = \{ x \in C(J, \mathbb{R}) \mid x \geq \theta \} \) which is a closed set in \( C(J, \mathbb{R}) \).

**4. Special cases and example**

1. When \( f(t, x, y) = 1 \) for all \( t \in J \) and \( x, y \in \mathbb{R} \), the QDE (1.1) reduces to the following known nonlinear differential equation with maxima

\[
\begin{align*}
x'(t) + \lambda x(t) &= g(t, x(t), X(t)), \quad t \in J, \\
x(t_0) &= x_0 \in \mathbb{R}_+.
\end{align*}
\]

(4.1)

The above nonlinear differential equation with maxima (3.10) has already been discussed in the literature for existence and uniqueness of the solutions via classical methods of Schauder and Banach fixed point principles. See Bainov and Hristova [1] and the references therein. Here our method is different and constructive. Therefore, Theorem 3.1 includes the existence and approximation theorem for the differential equation with maxima (3.10) as a special case under weak partial compactness type conditions.

2. Again, when \( f(t, x, y) = f(t, x) \) and \( g(t, x, y) = g(t, y) \) for all \( t \in J \) and \( x, y \in \mathbb{R} \), the QDE (1.1) reduces to the following QDE with maxima,

\[
\begin{align*}
\frac{d}{dt} \left[ \frac{x(t)}{f(t, x(t))} \right] + \lambda \left[ \frac{x(t)}{f(t, x(t))} \right] &= g(t, X(t)), \quad t \in J, \\
x(t_0) &= x_0 \in \mathbb{R}_+,
\end{align*}
\]

(4.2)

which can be discussed in as in Dhage and Dhage [9] via Dhage iteration method and established the existence and approximation result for positive solutions.
3. If \( f(t, x, y) = f(t, y) \) and \( g(t, x, y) = g(t, y) \) for all \( t \in J \) and \( x, y \in \mathbb{R} \), then QDE (1.1) reduces to the following QDE with maxima,

\[
\begin{align*}
\frac{d}{dt} \left[ \frac{x(t)}{f(t, X(t))} \right] + \lambda \left[ \frac{x(t)}{f(t, X(t))} \right] = g(t, X(t)), \quad t \in J, \\
x(t_0) = x_0 \in \mathbb{R}_+,
\end{align*}
\]

which is also new but could also be discussed under same hypotheses and arguments that given in Dhage and Dhage [9] without any change via Dhage iteration method for the existence and approximation result of positive solutions.

4. If we take \( f(t, x, y) = py + F(t) \) for all \( t \in J \) and \( x, y \in \mathbb{R} \) in (4.1), then it reduces to the standard linear differential equation of automatic regulation,

\[
\begin{align*}
x'(t) + \lambda x(t) = pX(t) + F(t), \quad t \in J, \\
x(t_0) = x_0 \in \mathbb{R}_+,
\end{align*}
\]

for all \( t \in J \), where \( \lambda > 0, p > 0 \) are constants and \( F : J \to \mathbb{R} \) is a continuous perturbation function. The differential equation with maxima (4.4) is the motivation for development of the subject of differential equations with maxima. Therefore, our QDE (1.1) is more general and Theorem 3.1 includes the existence and approximation results for all the above differential equations with maxima as special cases.

Finally we give an example to illustrate the hypotheses and the main abstract result formulated in Theorem 3.1.

**Example 4.1.** Given a closed and bounded interval \( J = [0, 1] \), consider the IVP of QDE,

\[
\begin{align*}
\frac{d}{dt} \left[ \frac{x(t)}{f(t, x(t), X(t))} \right] = \frac{1}{9} \left[ 2 + \tanh x(t) + \tanh X(t) \right], \quad t \in J, \\
x(0) = 0,
\end{align*}
\]

where the functions \( f, g : J \times \mathbb{R} \to \mathbb{R} \) are defined as

\[
f(t, x, y) = \begin{cases} 
1, & \text{if } x \leq 0, \\
1 + x + y, & \text{if } 0 < x, y < 3, \\
7, & \text{if } x \geq 3, y \geq 3,
\end{cases}
\]

and

\[
g(t, x, y) = \frac{1}{9} \left[ 2 + \tanh x + \tanh y \right].
\]

Clearly, the functions \( f \) and \( g \) are continuous on \( J \times \mathbb{R} \times \mathbb{R} \) into \( \mathbb{R}_+ \setminus \{0\} \).

As \( \frac{\partial}{\partial x} \left( \frac{x}{f(t, x, x)} \right) \geq 0 \) for all \( t \in J \), the function \( x \mapsto \frac{x}{f(t, x, x)} \) is increasing for each \( t \in J \) and so, the hypothesis \( (A_0) \) is satisfied in view of Remark 3.1.

Next, \( f \) satisfies the hypothesis \( (A_3) \) with \( \varphi(r) = 2r \). To see this, we have

\[0 \leq f(t, x_1, x_2) - f(t, y_1, y_2) \leq 2 \max\{x_1 - y_1, x_2 - y_2\}\]
Dhage iteration method for quadratic differential equations

for all \( x_1 \geq y_1, x_2 \geq y_2 \). Therefore, \( \varphi(r) = 2r \). Moreover, the function \( f(t, x, y) \) is positive and bounded on \( J \times \mathbb{R} \times \mathbb{R} \) with bound \( M_f = 7 \) and so the hypothesis (A2) is satisfied. Again, since \( g \) is positive and bounded on \( J \times \mathbb{R} \times \mathbb{R} \) by \( M_g = \frac{4}{9} \), the hypothesis (B2) holds. Furthermore, \( g(t, x, y) \) is nondecreasing in \( x \) and \( y \) for all \( t \in J \), and thus hypothesis (B3) is satisfied. Also condition (3.4) of Theorem 3.1 is held. Finally, the QDE (4.5) has a lower solution \( u(t) = \frac{t}{9} \) defined on \( J \). Thus all hypotheses of Theorem 3.1 are satisfied. Hence we apply Theorem 3.1 and conclude that the QDE (4.4) has a positive solution \( x^* \) defined on \( J \) and the sequence \( \{x_n\}_{n=1}^{\infty} \) defined by

\[
x_{n+1}(t) = \frac{1}{9} [f(t, x_n(t), X_n(t))] \left( \int_0^t [2 + \tanh x_n(s) + \tanh X_n(s)] \, ds \right),
\]

for all \( t \in J \), where \( x_1 = u \), converges monotonically to \( x^* \).

References


