Nonlinear Analysis Forum **21**(1), pp. 77–86, 2016
Available electronically at http://www.na-forum.org

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Reprinted from the
Nonlinear Analysis Forum
Vol. 21(1), Apr. 2016

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ON HADAMARD INEQUALITIES FOR RELATIVE CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this paper we prove Hadamard inequalities for relative convex functions via k-fractional Riemann-Liouville integrals and Hadamard inequalities for relative convex function via fractional Riemann-Liouville integrals are deduced.

1. Introduction

Fractional calculus is a branch of mathematics that developed with classical definitions of integral and derivatial calculus. The fundamental concepts of fractional calculus were developed by the famous mathematicians Leibniz, Liouville and Riemann centuries ago. This field caught the attention of engineers in 1890. Topical monographs and symposia procedures have tinted the appliance of fractional calculus in physics, continuum mechanics, signal processing, and electromagnetic [2].

Convex functions provide a basis for defining a lot of new functions and related inequalities for example Noor define relative convex function on a relative convex set T_g [13].

A function $f : [a, b] \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

Received & Accepted: Mar. 2016, Online Published: Apr. 2016.
2010 Mathematics Subject Classification: Primary 26A51, 26A33; Secondary 26D10.
Key words and phrases: Relative convex functions, Hermite-Hadamard inequalities, Fractional integrals.

holds, for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. If $-f$ is convex, then f is called concave function and vice versa.

Consider a set T_g in $\mathbb{R}$. This set T_g is relative convex with respect to an arbitrary function $g : \mathbb{R} \to \mathbb{R}$ if it satisfies the following condition

$$(1-t)u + tg(v) \in T_g,$$

where $u, v \in \mathbb{R}$ such that $u, g(v) \in T_g$,$t \in [0, 1]$[10].

A function $f : T_g \to \mathbb{R}$ is said to be relative convex, if there exists an arbitrary function $g : \mathbb{R} \to \mathbb{R}$ such that

$$f((1-t)u + tg(v)) \leq (1-t)f(u) + tf(g(v)),$$

where $u, v \in \mathbb{R}$ such that $u, g(v) \in T_g$ and $t \in [0, 1]$ [13].

Every convex function is relative convex, but the converse is not true. For further details see [11, 12, 13].

In literature double integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, is known as Hadamard inequality. If f is concave then the above inequalities hold in the reverse direction. The Hadamard inequality got the attention of many mathematicians and many generalizations and refinements have been found so far for details see, [3, 4, 5, 7, 14, 16, 17] and references therein.

Let $f \in L_1[a, b]$. Then Riemann-Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined as follows:

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a \tag{1.2}$$

and

$$I^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b. \tag{1.3}$$

For details see [6, 15].

In [1] Díaz et al. defined k-Gamma function as follows.

If $k > 0$, then k-Gamma function Γ_k is defined as:

$$\Gamma_k(\alpha) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{\alpha}{k}} - 1}{(\alpha)_{n,k}}.$$

k-Gamma function in integral form is defined as

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-\frac{t^k}{k}}dt,$$

with the property that

$$\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha).$$

In [8] k-fractional Riemann-Liouville integrals are defined as follows:

Let $f \in L_1[a,b]$. Then k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as

$$I_{a+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a \tag{1.4}$$

and

$$I_{b-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b, \tag{1.5}$$

where

$$I_{a+}^{0,1} f(x) = I_{b-}^{0,1} f(x) = f(x).$$

For $k = 1$, k-fractional integrals give Riemann-Liouville integrals.

In [10], Noor et al. proved following Hadamard and Hadamard- type inequalities for relative convex functions via Riemann-Liouville fractional integrals.

Theorem 1. *Let f be a positive and relative convex function and $f \in L[a, g(b)]$. Then we have the following inequality*

$$f\left(\frac{a+g(b)}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(g(b)-a)^{\alpha}} \left[I_{a+}^{\alpha} f(g(b)) + I_{g(b)-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(g(b))}{2}. \tag{1.6}$$

Theorem 2. *Let $f : T_g \to \mathbb{R}$ be a differential mapping on interior of T_g and $f' \in L[a, g(b)]$. If $|f'|$ is relative convex on T_g, then*

$$\left|\frac{f(a)+f(g(b))}{2} - \frac{\Gamma(\alpha)+1}{2(g(b)-a)^{\alpha}}[I_{a+}^{\alpha} f(g(b)) + I_{g(b)-}^{\alpha} f(a)]\right| \leq \frac{g(b)-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[|f'(a)|+|f'(g(b))|\right]. \tag{1.7}$$

In this paper we generalize the fractional Hadamard and Hadamard-type inequalities (1.6) and (1.7) via k-fractional integrals and show that these inequalities are special cases of our results. Also we generalize the obtained results.

2. Hadamard inequalities for k-fractional integrals

First we give Hadamard and Hadamard-type inequalities for k-fractional Riemann-Liouville integrals.

Theorem 3. *Let $f : T_g \to \mathbb{R}$ be a positive function and $f \in L[a, g(b)]$. If f is relative convex function on T_g. Then the following inequalities for k-fractional integrals hold:*

$$f\left(\frac{a+g(b)}{2}\right) \leq \frac{\Gamma_k(\alpha+k)}{2(g(b)-a)^{\frac{\alpha}{k}}} \left[I_{a+}^{\alpha,k} f(g(b)) + I_{g(b)-}^{\alpha,k} f(a)\right] \leq \frac{f(a)+f(g(b))}{2} \tag{2.1}$$

with $\alpha, k > 0$.

Proof. As f is relative convex on T_g therefore we have,

$$f\left(\frac{a+g(b)}{2}\right)=f\left(\frac{1}{2}(ta+(1-t)g(b))+\left(1-\frac{1}{2}\right)((1-t)a+tg(b))\right)$$
$$\leq \frac{1}{2}f\left(ta+(1-t)g(b)\right)+\left(1-\frac{1}{2}\right)f\left((1-t)a+tg(b)\right).$$

This implies

$$2f\left(\frac{a+g(b)}{2}\right)\leq f(ta+(1-t)g(b))+f((1-t)a+tg(b)),$$

multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$, and integrating over $[0,1]$ we have,

$$\frac{2k}{\alpha}f\left(\frac{a+g(b)}{2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}dt$$
$$\leq \int_0^1 t^{\frac{\alpha}{k}-1}f(ta+(1-t)g(b))dt+\int_0^1 t^{\frac{\alpha}{k}-1}f((1-t)a+tg(b))dt$$
$$=\frac{k\Gamma_k(\alpha)}{(g(b)-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k}f(g(b))+I_{g(b)-}^{\alpha,k}f(a)\right],$$

from which one can have

$$f\left(\frac{a+g(b)}{2}\right)\leq \frac{\Gamma_k(\alpha+k)}{2(g(b)-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k}f(g(b))+I_{g(b)-}^{\alpha,k}f(a)\right]. \tag{2.2}$$

On the other hand relative convexity of f on T_g gives,

$$f(ta+(1-t)g(b))+f((1-t)a+tg(b))\leq tf(a)+(1-t)f(g(b))+(1-t)f(a)+tf(g(b)),$$

multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$, and integrating over $[0,1]$ we have,

$$\int_0^1 t^{\frac{\alpha}{k}-1}f(ta+(1-t)g(b))dt+\int_0^1 t^{\frac{\alpha}{k}-1}f((1-t)a+tg(b))dt$$
$$\leq [f(a)+f(g(b))]\int_0^1 t^{\frac{\alpha}{k}-1}dt,$$

from which one can have

$$\frac{\Gamma_k(\alpha+k)}{2(g(b)-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k}f(g(b))+I_{g(b)-}^{\alpha,k}f(a)\right]\leq \frac{f(a)+f(g(b))}{2}. \tag{2.3}$$

Combining inequality (2.2) and inequality (2.3) we get inequality (2.1).

For next result we need the following lemma.

Lemma 1. *Let* $f:T_g\to\mathbb{R}$ *be a differentiable mapping on interior of* T_g. *If* $f^{'}\in L[a,g(b)]$, *then the following equality for k-fractional integral holds:*

$$\frac{f(a)+f(g(b))}{2}-\frac{\Gamma_k(\alpha+k)}{2(g(b)-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k}f(g(b))+I_{g(b)-}^{\alpha,k}f(a)\right]$$
$$=\frac{g(b)-a}{2}\int_0^1((1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}})f^{'}(ta+(1-t)g(b))dt. \quad (2.4)$$

Proof. One can note that

$$\frac{g(b)-a}{2}\int_0^1((1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}})f^{'}(ta+(1-t)g(b))dt$$
$$=\frac{g(b)-a}{2}\left[\int_0^1(1-t)^{\frac{\alpha}{k}}f^{'}(ta+(1-t)g(b))dt-\int_0^1 t^{\frac{\alpha}{k}}f^{'}(ta+(1-t)g(b))dt\right],$$

where by simple calculation one can get

$$\int_0^1(1-t)^{\frac{\alpha}{k}}f^{'}(ta+(1-t)g(b))dt$$
$$=\frac{f(g(b))}{g(b)-a}-\frac{\alpha}{k(g(b)-a)}\int_a^{g(b)}\left(\frac{x-a}{g(b)-a}\right)^{\frac{\alpha}{k}-1}\frac{f(x)}{g(b)-a}dx$$
$$=\frac{f(g(b))}{g(b)-a}-\frac{\Gamma_k\left(\alpha+k\right)}{(g(b)-a)^{\frac{\alpha}{k}+1}}I_{g(b)-}^{\alpha,k}f(a)$$

and

$$-\int_0^1 t^{\frac{\alpha}{k}}f^{'}(ta+(1-t)g(b))dt$$
$$=\frac{f(a)}{g(b)-a}-\frac{\alpha}{k(g(b)-a)}\int_a^{g(b)}\left(\frac{g(b)-x}{g(b)-a}\right)^{\frac{\alpha}{k}-1}\frac{f(x)}{g(b)-a}dx$$
$$=\frac{f(a)}{g(b)-a}-\frac{\Gamma_k\left(\alpha+k\right)}{(g(b)-a)^{\frac{\alpha}{k}+1}}I_{a+}^{\alpha,k}f(g(b)).$$

Hence (2.4) can be obtained.

Using above lemma we give the following k-fractional Hadamard-type inequality.

Theorem 4. *Let $f:T_g\to\mathbb{R}$ be a differentiable mapping on interior of T_g and $f'\in L[a,g(b)]$. If $|f'|$ is relative convex on T_g, then the following inequality for k-fractional integral holds:*

$$\left|\frac{f(a)+f(g(b))}{2}-\frac{\Gamma_k(\alpha+k)}{2(g(b)-a)^{\frac{\alpha}{k}}}\left[I_{a+}^{\alpha,k}f(g(b))+I_{g(b)-}^{\alpha,k}f(a)\right]\right|$$
$$\leq\frac{g(b)-a}{2(\frac{\alpha}{k}+1)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left[|f'(a)|+|f'(g(b))|\right] \quad (2.5)$$

with $\alpha,k>0$.

Proof. From Lemma 1 and relative convexity of $|f'|$ on T_g, we have,

$$\begin{aligned}
&\left|\frac{f(a)+f(g(b))}{2}-\frac{\Gamma_k(\alpha+k)}{2(g(b)-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{a+}f(g(b))+I^{\alpha,k}_{g(b)-}f(a)\right]\right|\\
&\leq \frac{g(b)-a}{2}\int_0^1\left|(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|\left|f'(ta+(1-t)g(b))\right|dt\\
&\leq \frac{g(b)-a}{2}\int_0^1|(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}|[t|f'(a)|+(1-t)|f'(g(b))|]dt\\
&= \frac{g(b)-a}{2}\left[\int_0^{\frac{1}{2}}((1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}})(t\left|f'(a)\right|+(1-t)|f'(g(b))|)\right.\\
&\left.+\int_{\frac{1}{2}}^1\left[t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}}\right][t|f'(a)|+(1-t)|f'(g(b))|]dt\right]\\
&= \frac{g(b)-a}{2}(I_1+I_2). \qquad (2.6)
\end{aligned}$$

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}}[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}][t|f'(a)|+(1-t)|f'(g(b))|dt\\
&= |f'(a)|\left[\int_0^{\frac{1}{2}}t(1-t)^{\frac{\alpha}{k}}dt-\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}+1}dt\right]\\
&+|f'(g(b))|\left[\int_0^{\frac{1}{2}}(1-t)^{\frac{\alpha}{k}+1}-\int_0^{\frac{1}{2}}(1-t)t^{\frac{\alpha}{k}}dt\right]\\
&= |f'(a)|\left[\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]+|f'(g(b))|\left[\frac{1}{\frac{\alpha}{k}+2}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right],
\end{aligned}$$

and by similar evaluation we have

$$I_2 = |f'(a)|\left[\frac{1}{\frac{\alpha}{k}+2}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]+|f'(g(b))|\left[\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right].$$

Therefore (2.6) implies,

$$\begin{aligned}
&\left|\frac{f(a)+f(g(b))}{2}-\frac{\Gamma_k(\alpha+k)}{2(g(b)-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{a+}f(g(b))+I^{\alpha,k}_{g(b)-}f(a)\right]\right|\\
&\leq \frac{g(b)-a}{2}\left[\left|f'(a)\right|\left[\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]+\left|f'(g(b))\right|\left[\frac{1}{\frac{\alpha}{k}+2}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]\right.\\
&\left.+\left|f'(a)\right|\left[\frac{1}{\frac{\alpha}{k}+2}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]+|f'(g(b))|\left[\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}-\frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]\right].
\end{aligned}$$

From which after a little computation one can have (2.5).

Remark 1. If we take $k = 1$ in Theorem 4, we get inequality (1.7).

Next we give generalization of above obtained results.

Theorem 5. *Let $f : T_g \to \mathbb{R}$ be a positive function and $f \in L[g(a), g(b)]$. If f is relative convex function on T_g, then the following inequalities for generalized k-fractional integrals hold:*

$$f\left(\frac{g(a)+g(b)}{2}\right) \leq \frac{\Gamma_k(\alpha+k)}{2(g(b)-g(a))^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{g(a)+}f(g(b)) + I^{\alpha,k}_{g(b)-}f(a)\right] \leq \frac{f(g(a))+f(g(b))}{2} \tag{2.7}$$

with $\alpha, k > 0$.

Proof. By the relative convexity of f on T_g we have,

$$2f\left(\frac{g(a)+g(b)}{2}\right) \leq f(tg(a)+(1-t)g(b)) + f((1-t)g(a)+tg(b)),$$

multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$, and integrating over $[0, 1]$ we have,

$$\begin{aligned}
&\frac{2k}{\alpha} f\left(\frac{g(a)+g(b)}{2}\right) \int_0^1 t^{\frac{\alpha}{k}-1} dt \\
&\leq \int_0^1 t^{\frac{\alpha}{k}-1} f(tg(a)+(1-t)g(b))dt + \int_0^1 t^{\frac{\alpha}{k}-1} f((1-t)g(a)+tg(b))dt \\
&= \frac{k\Gamma_k(\alpha)}{(g(b)-g(a))^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{g(a)+}f(g(b)) + I^{\alpha,k}_{g(b)-}f(g(a))\right],
\end{aligned}$$

from which one can have

$$f\left(\frac{g(a)+g(b)}{2}\right) \leq \frac{\Gamma_k(\alpha+k)}{2(g(b)-g(a))^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{g(a)+}f(g(b)) + I^{\alpha,k}_{g(b)-}f(g(a))\right]. \tag{2.8}$$

On the other hand relative convexity of f on T_g gives,

$$\begin{aligned}
&f(tg(a)+(1-t)g(b)) + f((1-t)g(a)+tg(b)) \\
&\leq tf(g(a)) + (1-t)f(g(b)) + (1-t)f(g(a)) + tf(g(b)),
\end{aligned}$$

multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$, and integrating over $[0, 1]$ we have,

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}-1} f(tg(a)+(1-t)g(b))dt + \int_0^1 t^{\frac{\alpha}{k}-1} f((1-t)g(a)+tg(b))dt \\
&\leq [f(g(a))+f(g(b))] \int_0^1 t^{\frac{\alpha}{k}-1} dt,
\end{aligned}$$

from which one can have

$$\frac{\Gamma_k(\alpha+k)}{2(g(b)-g(a))^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{g(a)+}f(g(b))+I^{\alpha,k}_{g(b)-}f(g(a))\right]\leq\frac{f(g(a))+f(g(b))}{2}. \tag{2.9}$$

Combining inequality (2.8) and inequality (2.9) we get inequality (2.7) .

Corollary 1. *If we take $k=1$ in above theorem we get the following inequality*

$$f\left(\frac{g(a)+g(b)}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(g(b)-g(a))^{\alpha}}\left[I^{\alpha}_{g(a)+}f(g(b))+I^{\alpha}_{g(b)-}f(a)\right]\leq\frac{f(g(a))+f(g(b))}{2} \tag{2.10}$$

with $\alpha>0$.

For next result we need the following lemma.

Lemma 2. *Let $f:T_g\to\mathbb{R}$ be a differentiable mapping on interior of T_g. If $f^{'}\in L[g(a),g(b)]$, then the following equality for k-fractional integral holds:*

$$\frac{f(g(a))+f(g(b))}{2}-\frac{\Gamma_k(\alpha+k)}{2(g(b)-g(a))^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{g(a)+}f(g(b))+I^{\alpha,k}_{g(b)-}f(g(a))\right]=\frac{g(b)-g(a)}{2}\int_0^1((1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}})f^{'}(tg(a)+(1-t)g(b))dt. \tag{2.11}$$

Proof. Proof follows from Lemma 1.

Using above lemma we give the following k-fractional Hadamard-type inequality.

Theorem 6. *Let $f:T_g\to\mathbb{R}$ be a differentiable mapping on interior of T_g and $f'\in L[g(a),g(b)]$. If $|f'|$ is relative convex on T_g, then the following inequality for k-fractional integral holds:*

$$\left|\frac{f(g(a))+f(g(b))}{2}-\frac{\Gamma_k(\alpha+k)}{2(g(b)-g(a))^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{g(a)+}f(g(b))+I^{\alpha,k}_{g(b)-}f(g(a))\right]\right|\leq\frac{g(b)-g(a)}{2(\frac{\alpha}{k}+1)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left[|f'(g(a))|+|f'(g(b))|\right] \tag{2.12}$$

with $\alpha,k>0$.

Proof. From Lemma 2 and relative convexity of $|f'|$ on T_g, we have,

$$\left|\frac{f(g(a))+f(g(b))}{2}-\frac{\Gamma_k(\alpha+k)}{2(g(b)-g(a))^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{g(a)+}f(g(b))+I^{\alpha,k}_{g(b)-}f(g(a))\right]\right|\leq\frac{g(b)-g(a)}{2}\int_0^1\left|(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|\left|f^{'}(tg(a)+(1-t)g(b))\right|dt$$

$$\leq \frac{g(b)-g(a)}{2}\int_0^1 |(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}|[t|f'(g(a))| + (1-t)|f'(g(b))|]dt$$

$$= \frac{g(b)-g(a)}{2}\Bigg[\int_0^{\frac{1}{2}} ((1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}})(t\left|f'(g(a))\right| + (1-t)|f'(g(b))|)$$

$$+ \int_{\frac{1}{2}}^1 \left[t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}}\right][t|f'(g(a))| + (1-t)|f'(g(b))|]dt\Bigg]$$

$$= \frac{g(b)-g(a)}{2}(I_1 + I_2). \tag{2.13}$$

$$I_1 = \int_0^{\frac{1}{2}} [(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}][t|f'(g(a))| + (1-t)|f'(g(b))|dt$$

$$= |f'(g(a))|\left[\int_0^{\frac{1}{2}} t(1-t)^{\frac{\alpha}{k}}dt - \int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}+1}dt\right]$$

$$+ |f'(g(b))|\left[\int_0^{\frac{1}{2}} (1-t)^{\frac{\alpha}{k}+1} - \int_0^{\frac{1}{2}} (1-t)t^{\frac{\alpha}{k}}dt\right]$$

$$= |f'(g(a))|\left[\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)} - \frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right] + |f'(g(b))|\left[\frac{1}{\frac{\alpha}{k}+2} - \frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]$$

and by similar evaluation we have

$$I_2 = |f'(g(a))|\left[\frac{1}{\frac{\alpha}{k}+2} - \frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right] + |f'(g(b))|\left[\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)} - \frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right].$$

Therefore (2.13) implies,

$$\left|\frac{f(g(a)) + f(g(b))}{2} - \frac{\Gamma_k(\alpha+k)}{2(g(b)-g(a))^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{g(a)+}f(g(b)) + I^{\alpha,k}_{g(b)-}f(g(a))\right]\right|$$

$$\leq \frac{g(b)-g(a)}{2}\Bigg[\left|f'(g(a))\right|\left[\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)} - \frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]$$

$$+ \left|f'(g(b))\right|\left[\frac{1}{\frac{\alpha}{k}+2} - \frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right] + \left|f'(g(a))\right|\left[\frac{1}{\frac{\alpha}{k}+2} - \frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]$$

$$+|f'(g(b))|\left[\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)} - \frac{(\frac{1}{2})^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right]\Bigg].$$

From which after a little computation one can have (2.12).

Corollary 2. *For $k = 1$ in Theorem 6 we get following inequality.*

$$\left|\frac{f(g(a))+f(g(b))}{2}-\frac{\Gamma(\alpha+1)}{2(g(b)-g(a))^{\alpha}}\left[I^{\alpha}_{g(a)+}f(g(b))+I^{\alpha}_{g(b)-}f(g(a))\right]\right| \leq \frac{g(b)-g(a)}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[|f'(g(a))|+|f'(g(b))|\right], \quad (2.14)$$

with $\alpha > 0$.

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