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ON THE EXISTENCE OF UNBOUNDED ENDOGENOUS ECONOMIC GROWTH

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Abstract. We study the existence of unbounded knowledge-based economic growth under a general technological structure. When unbounded growth is feasible, we provide estimates for the growth rate.

1. Introduction

The origin of endogenous growth literature can be traced back to Arrow's (1962) learning-by-doing approach, Uzawa's (1965) 2-sector model with an education sector and Shell (1966) treatment of education investment as consuming income - all were early attempts to endogenize Solow's (1956, 1957) residuals. The recent literature follows Romer (1986, 1990), Lucas (1988), Grossman and Helpman (1991) and Aghion and Howitt (1992).[1] These models establish balanced (exponential) growth without resorting to exogenous processes by imposing particular structures on the underlying economy. For example, in Lucas's (1988) model sustained growth is attained by labor-augmenting human capital which is produced by labor and the existing stock of human capital in a very particular form, and in Romer's (1990) model sustained growth is achieved when the stock of capital consists of varieties of intermediate goods such that an increase in the capital stock takes the form of an increase in the number of varieties by a deliberate R&D process that, again, takes a particular form. In both cases (and in other endogenous growth models) the key step is to overcome the constraining effects of diminishing marginal productivity of capital: in the former model this is

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[1]See Solow (1994, 2000) for an overview.

accomplished by the human capital that augments labor productivity, and in the latter by allowing capital to increase by expanding the number of varieties.

A characteristic feature of these models (and of other endogenous growth models) is that convergence to a balanced (exponential) growth depends on a finely tuned parameter value, such that a small deviation results in the economy either growing too fast, reaching infinity at a finite time, or too slow, eventually converging to a stationary steady state. This "knife-hedge" property (as it was called by Solow 1994, 2000) is disturbing because it appears to rely on an arbitrary assumption and undermines the robustness of balanced growth as a long-run property in endogenous growth theory. In particular, it raises a question regarding the feasibility of unbounded (exponential or otherwise) endogenous growth.

This work addresses this question. We consider a general economic structure, by imposing minimal constraints, and study the conditions under which unbounded growth is feasible. The underlying technology is loosely specified (requiring it to satisfy very mild conditions) and the growth mechanism considered is that of labor-augmenting human capital (as in Lucas 1988) that competes with ordinary capital and consumption for the available resources (output). When unbounded growth is feasible, we provide lower bound on the long-run rate of growth in terms of properties of the labor-augmenting and production functions.

The next section describes the economy in terms of a production technology, a labor-augmenting technical change function and a budget constraint. The economic structure is general in that very mild assumptions are imposed on the production technology and the technical change process. Section 3 presents our main results regarding the existence of unbounded growth, as well as estimates of growth rates, in the form of two theorems, which are proven in Sections 5 and 6. Auxiliary material needed to prove the main theorems is presented in Section 4 and Section 7 concludes the paper.

2. The economy

The economy is characterized by a production technology and by a budget constraint. The technology is very loosely defined, imposing minimal structure. The budget constraint is just a material balance requirement, identifying feasible investments (in both types of capital) and consumption actions.

The state of the economy at time t (here $t \geq 0$ is an integer) is a triplet (K_t, h_t, L_t) where K_t is capital, h_t is human-capital and L_t is labor. Then the production value at moment $t+1$ is $F(K_t, A(h_t)L_t)$, where $F(\cdot,\cdot)$ is a production function and the function $A(\cdot)$ measures the influence of the human-capital. To simplify we assume constant labor $L_t = L_0$ (a standard assumption in the theory of the economic growth), so per capita output takes the form

$$y_t \equiv L_0^{-1} F(K_t, A(h_t)L_0) = A(h_t) f(K_t L_0^{-1} A(h_t)^{-1}), \tag{2.1}$$

where $f(x) = F(x,1)$ for all $x \geq 0$. In other words, for any $x \geq 0$, $f(x)$ is the value production at time t, when the capital at time t is x and the human-capital h_t and labor L_t satisfy $A(h_t)L_t = 1$.

Let $v \in [0,1)$ represent the depreciation rate of K (we assume human capital does not depreciate), so

$$K_{t+1} \geq vK_t,\ h_{t+1} \geq h_t. \tag{2.2}$$

The following minimal assumptions about $f(\cdot)$ and $A(\cdot)$ are maintained:

Assumption 2.1. $f : [0,\infty) \to [0,\infty)$ *is increasing, continuous, concave, satisfies* $f(0) = 0$ *and there exists* $x^* > 0$ *such that*

$$f(x) > (1-v)x \text{ for all } x \in (0, x^*) \tag{2.3}$$

and

$$f(x) < (1-v)x \text{ for all } x > x^*. \tag{2.4}$$

$A : [0,\infty) \to [0,\infty)$ *is increasing and*

$$A(h) > 0 \text{ for all } h > 0. \tag{2.5}$$

The economy's budget constraint at time t is

$$K_{t+1} - vK_t + C_t + L_0(h_{t+1} - h_t) \leq F(K_t, A(h_t)L_0),$$

where $C_t \geq 0$ denote aggregate consumption. In terms of the per capita quantities $k_t = L_0^{-1}K_t$ and $c_t = L_0^{-1}C_t$, the budget constraint can be rendered as

$$k_{t+1} - vk_t + c_t + h_{t+1} - h_t \leq A(h_t)f(k_tA(h_t)^{-1}). \tag{2.6}$$

We proceed now to study growth prospects of an economy characterized by the functions $f(\cdot)$ and $A(\cdot)$, satisfying Assumption 2.1. Note that the concavity assumption on f is a usual assumption in the economic growth theory.

3. Existence of unbounded growth and balanced growth estimates

Let $T_1 \geq 0$ and $T_2 > T_1$ be two integers. The triplet $(\{k_t\}_{t=T_1}^{T_2}, \{h_t\}_{t=T_1}^{T_2}, \{c_t\}_{t=T_1}^{T_2-1})$ is called a trajectory if

$$k_t \geq 0,\ t = T_1, \dots, T_2,\ h_t > 0,\ t = T_1, \dots, T_2,\ c_t \geq 0,\ t = T_1, \dots, T_2 - 1$$

and for all integers t satisfying $T_1 \leq t < T_2$, the budget constraint (2.6) and

$$k_{t+1} \geq vk_t,\ h_{t+1} \geq h_t \tag{3.1}$$

are satisfied. A triplet $(\{k_t\}_{t=0}^{\infty}, \{h_t\}_{t=0}^{\infty}, \{c_t\}_{t=0}^{\infty})$ is called a trajectory if the triplet

$$(\{k_t\}_{t=0}^{T}, \{h_t\}_{t=0}^{T}, \{c_t\}_{t=0}^{T-1})$$

is a trajectory for any natural number T.

The question that concerns us in this work is the following: can a technology, characterized by the mild conditions stated in Assumption 2.1, support unbounded growth? Or, more precisely, does there exist a trajectory

$(\{k_t\}_{t=0}^{\infty}, \{h_t\}_{t=0}^{\infty}, \{c_t\}_{t=0}^{\infty})$ such that $\lim_{t\to\infty} k_t = \infty$, $\lim_{t\to\infty} c_t = \infty$ and $\lim_{t\to\infty} h_t = \infty$? We will answer this question in the affirmative and provide bounds for the growth rate. The underlying driving force, of course, is human capital. We thus consider first the case in which human capital is constant, i.e., the classical one-sector model.

Notice, from (2.3)-(2.4), that

$$f(x^*) = (1-v)x^*. \tag{3.2}$$

Define

$$g(x) = vx + f(x),\ x \in [0,\infty), \tag{3.3a}$$

$$g^0(x) = x,\ x \in [0,\infty) \tag{3.3b}$$

and

$$g^1 = g,\ g^{t+1} = g^t \circ g \text{ for all integers } t \geq 0. \tag{3.3c}$$

The following result follows from (2.3), (2.4), (3.2) and (3.3).

Proposition 3.1. *For all* $x > 0$,

$$\lim_{t\to\infty} g^t(x) = x^*$$

and the sequence $\{g^t(x)\}_{t=0}^{\infty}$ *is strictly decreasing if* $x > x^*$ *and strictly increasing if* $x \in (0, x^*)$.

Suppose that human capital is constant at $h_0 > 0$ and $A(h_0) = 1$. For this submodel, trajectories are pairs of sequences $(\{k_t\}_{t=0}^{\infty}, \{c_t\}_{t=0}^{\infty})$ such that, for all integers $t \geq 0$,

$$k_t \geq 0,\ c_t \geq 0,\ \ k_{t+1} \geq vk_t$$

and

$$k_{t+1} - vk_t + c_t \leq f(k_t).$$

Proposition 3.1, then, implies that all trajectories of this submodel are bounded, in that the long-run upper limit of consumption and capital does not exceed x^*.

We now relax the assumption that h is constant and obtain the following property:

Theorem 3.2. *For each* $h_0 > 0$ *and each* $k_0 > 0$, *there exists a trajectory*

$$(\{k_t\}_{t=0}^{\infty}, \{h_t\}_{t=0}^{\infty}, \{c_t\}_{t=0}^{\infty})$$

such that

$$\lim_{t\to\infty} h_t = \infty.$$

If in addition $\lim_{x\to\infty} A(x) = \infty$, *then*

$$\lim_{t\to\infty} k_t = \infty \ \ \textit{and} \ \ \lim_{t\to\infty} c_t = \infty.$$

The next result provides lower bounds on the growth rate of capital and consumption under certain conditions.

Theorem 3.3. *Let $h_0 > 0$, $k_0 > 0$ and*

$$\sup\left\{\frac{A(t+1)}{A(t)}:\ t \in [h_0, \infty)\right\} < \infty.$$

Then there exist a natural number q_0, a positive number $\bar{c}$ and a trajectory

$$(\{k_t\}_{t=0}^{\infty}, \{h_t\}_{t=0}^{\infty}, \{c_t\}_{t=0}^{\infty})$$

such that for all natural numbers $t \geq q_0$,

$$h_t \geq tq_0^{-1},\ k_{t+1} \geq \bar{c}A(tq_0^{-1}),\ c_t \geq \bar{c}A(tq_0^{-1}).$$

If $A(\cdot)$ is bounded below by a function that increases at the rate $\alpha > 0$, then the following can be shown:

Corollary 3.4. *Let the assumptions of Theorem 3.3 hold, for some $\alpha > 0$*

$$A(x) \geq x^{\alpha} \text{ for all } x > 0$$

and let a natural number q_0, a positive number $\bar{c}$ and a trajectory

$$(\{k_t\}_{t=0}^{\infty}, \{h_t\}_{t=0}^{\infty}, \{c_t\}_{t=0}^{\infty})$$

be as guaranteed by Theorem 3.3. Then for all natural numbers $t \geq q_0$,

$$k_{t+1} \geq \bar{c}(tq_0^{-1})^{\alpha},\ c_t \geq \bar{c}(tq_0^{-1})^{\alpha}.$$

Theorems 3.2 and 3.3 follow, respectively, from Theorems 3.6 and 3.5, which are stated below. These theorems, in addition to establishing existence, provide uniform estimates on the growth rate of capital and consumption for all initial states $h_0 \geq \bar{h}$, $k_0 \geq \bar{A}(h_0)\Delta$, where $\bar{h}, \Delta$ are arbitrary positive numbers. We use the following notation: If $x \in R$, $x = n + r$, where n is an integer and $r \in [0, 1)$, then $[x] = n$.

Theorem 3.5. *Let*

$$\bar{h} \in (0,1),\quad \Lambda_0 > 1$$

$$A(t+1)A(t)^{-1} \leq \Lambda_0 \text{ for all } t \geq \bar{h}$$

and let $\Delta \in (0, 1)$ satisfy

$$\Delta < x^*/8,$$

$$f(\Delta), \Delta < (8\Lambda_0)^{-1} f(x^*/8).$$

Assume that an integer $\tau_0 > 3$ satisfies

$$g^{\tau_0 - 1}(\Delta) > x^*/2,$$

a natural number q satisfies

$$4^{-1}qA(\bar{h})(f(x^*/2) \geq 1$$

and that

$$h_0 \geq \bar{h},\ k_0 \geq A(\bar{h})\Delta.$$

Then there exists a trajectory $(\{k_t\}_{t=0}^{\infty}, \{h_t\}_{t=0}^{\infty}, \{c_t\}_{t=0}^{\infty})$ such that for all natural numbers j,

$$k_{jq\tau_0} \geq A(\bar{h}+j)\Delta,\ h_{jq\tau_0} \geq h_0 + j,$$

$$c_t = 2^{-1}A(\bar{h}+j-1)(f(\Delta)-(1-v)\Delta),\ t=(j-1)q\tau_0,\dots,jq\tau_0-1,$$
$$k_t \ge 2^{-1}A(\bar{h}+j-1)f(\Delta),\ t=(j-1)q\tau_0+1,\dots,jq\tau_0.$$

Moreover, for all integers $t\ge 0$,

$$h_t \ge h_0 + [t(q\tau_0)^{-1}],$$
$$k_{t+1} \ge 2^{-1}A(\bar{h}+[t(q\tau_0)^{-1}])f(\Delta),$$
$$c_t = 2^{-1}A(\bar{h}+[t(q\tau_0)^{-1}])(f(\Delta)-(1-v)\Delta)$$

and for all integers $t\ge q\tau_0$,

$$h_t \ge t(q\tau_0)^{-1}-1,$$
$$k_{t+1} \ge 2^{-1}A(t(q\tau_0)^{-1}-1)f(\Delta),$$
$$c_t \ge 2^{-1}A(t(q\tau_0)^{-1}-1)(f(\Delta)-(1-v)\Delta).$$

Theorem 3.6. *Let*

$$\bar{h}\in(0,1),$$
$$\Delta\in(0,1),\ \Delta < x^*/8,$$
$$f(\Delta),\Delta < 8^{-1}f(x^*/8).$$

Assume that

$$h_0\ge\bar{h},\ k_0\ge A(h_0)\Delta.$$

Then there exists a trajectory $(\{k_t\}_{t=0}^\infty,\{h_t\}_{t=0}^\infty,\{c_t\}_{t=0}^\infty)$ *and a strictly increasing sequence of integers* $\{T_s\}_{s=0}^\infty$ *such that* $T_0=0$, *for all integers* $j\ge 0$,

$$h_{T_j}\ge h_0+j,\ k_{T_j}\ge A(h_{T_j})\Delta$$

and for all $t=T_j,\dots,T_{j+1}-1$,

$$c_t \ge 2^{-1}A(h_{T_j})\Delta,$$
$$k_t \ge 2^{-1}A(h_{T_j}),\ t=T_j+1,\dots,T_{j+1}.$$

4. Auxiliary results

We use the notation, definitions and assumptions introduced in Sections 2 and 3. Let

$$\bar{h}\in(0,1),\ \Delta\in(0,1) \tag{4.1}$$

satisfy

$$\Delta < x^*/8,\ f(\Delta),\Delta < f(x^*/8)/8. \tag{4.2}$$

Lemma 4.1. *Assume that for an integer* $s\ge 0$,

$$h_s\ge\bar{h}_s\ge\bar{h},\ k_s\ge A(\bar{h}_s)\Delta \tag{4.3}$$

and let for any integer $t\ge s$,

$$h_t=h_s,\ c_t=2^{-1}A(\bar{h}_s)(f(\Delta)-(1-v)\Delta), \tag{4.4}$$

$$k_{t+1}=vk_t+A(h_s)f(k_tA(h_s)^{-1})-c_t. \tag{4.5}$$

Then for all integers $t\ge s$,

$$(k_{t+1}-vk_t)(A(\bar{h}_s))^{-1}\ge 2^{-1}f(g^{t-s}(\Delta)). \tag{4.6}$$

Proof. Set
$$k_{1,s} = k_s,\ k_{2,s} = k_s \tag{4.7}$$
and for any integer $t \geq s$ set
$$k_{1,t+1} = vk_{1,t} + A(h_s)f(k_{1,t}A(h_s)^{-1}) - 2c_t, \tag{4.8}$$
$$k_{2,t+1} = vk_{2,t} + A(h_s)f(k_{2,t}A(h_s)^{-1}). \tag{4.9}$$
By (2.5), (4.1), (4.3) and concavity of f for all $x \geq 0$,
$$A(h_s)f(xA(h_s)^{-1}) = A(h_s)f(xA(\bar{h}_s)^{-1}(A(\bar{h}_s)A(h_s)^{-1}))$$
$$\geq A(h_s)A(\bar{h}_s)A(h_s)^{-1}f(xA(\bar{h}_s)^{-1}) = A(\bar{h}_s)f(xA(\bar{h}_s)^{-1}). \tag{4.10}$$
By (4.8), (4.9) and (4.10), for all integers $t \geq s$,
$$k_{1,t+1} \geq vk_{1,t} + A(\bar{h}_s)f(k_{1,t}A(\bar{h}_s)^{-1}) - 2c_t, \tag{4.11}$$
$$k_{2,t+1} \geq vk_{2,t} + A(\bar{h}_s)f(k_{2,t}A(\bar{h}_s)^{-1}). \tag{4.12}$$
Assume that an integer $t \geq s$ satisfies
$$k_{1,t} \geq A(\bar{h}_s)\Delta. \tag{4.13}$$
(Note that by (4.3) and (4.7) relation (4.13) holds with $t = s$.) By (4.4), (4.12) and (4.13),
$$k_{1,t+1} \geq vA(\bar{h}_s)\Delta + A(\bar{h}_s)f(\Delta) - A(\bar{h}_s)(f(\Delta) - (1-v)\Delta) = A(\bar{h}_s)\Delta.$$
Thus (4.13) holds for all integers $t \geq s$.

By (4.4), (4.11) and (4.13) for all integers $t \geq s$,
$$k_{1,t+1} - vk_{1,t} \geq A(\bar{h}_s)f(\Delta) - A(\bar{h}_s)(f(\Delta) - (1-v)\Delta)$$
$$\geq A(\bar{h}_s)(1-v)\Delta. \tag{4.14}$$
In view of (4.8),
$$k_{1,t+1} - vk_{1,t} + 2c_t \leq A(h_s)f(k_{1,t}A(h_s)^{-1}). \tag{4.15}$$
We show that for all integers $t \geq s$,
$$k_t \geq 2^{-1}(k_{1,t} + k_{2,t}). \tag{4.16}$$
By (4.7) inequality (4.16) holds for $t = s$.

Assume that an integer $t \geq s$ and that (4.16) holds. By (4.5), (4.8), (4.9), (4.16) and monotonicity and concavity of f,
$$k_{t+1} - 2^{-1}(k_{1,t+1} + k_{2,t+1})$$
$$= vk_t + A(h_s)f(k_tA(h_s)^{-1}) - c_t - 2^{-1}(vk_{1,t} + vk_{2,t})$$
$$-2^{-1}A(h_s)f(k_{1,t}A(h_s)^{-1}) + c_t - 2^{-1}A(h_s)f(k_{2,t}A(h_s)^{-1})$$
$$\geq A(h_s)[f(k_tA(h_s)^{-1}) - 2^{-1}f(k_{1,t}A(h_s)^{-1}) - 2^{-1}f(k_{2,t}A(h_s)^{-1})]$$
$$\geq A(h_s)[f(2^{-1}(k_{1,t}+k_{2,t})A(h_s)^{-1}) - 2^{-1}f(k_{1,t}A(h_s)^{-1}) - 2^{-1}f(k_{2,t}A(h_s)^{-1})] \geq 0.$$
Thus (4.16) holds for all integers $t \geq s$.

By (4.4), (4.5), (4.10), (4.13), (4.16) and monotonicity and concavity of f for all integers $t \geq s$,

$$k_{t+1} - vk_t \geq A(h_s)f(2^{-1}A(h_s)^{-1}(k_{1,t} + k_{2,t})) - c_t$$

$$\geq 2^{-1}A(h_s)f(A(h_s)^{-1}k_{1,t}) + 2^{-1}A(h_s)f(A(h_s)^{-1}k_{2,t}) - c_t$$

$$\geq 2^{-1}A(\bar{h}_s)f(A(\bar{h}_s)^{-1}k_{1,t}) + 2^{-1}A(\bar{h}_s)f(A(\bar{h}_s)^{-1}k_{2,t}) - c_t$$

$$\geq 2^{-1}A(\bar{h}_s)f(\Delta) - 2^{-1}A(\bar{h}_s)(f(\Delta) - (1-v)\Delta) + 2^{-1}A(\bar{h}_s)f(A(\bar{h}_s)^{-1}k_{2,t})$$

$$\geq 2^{-1}A(\bar{h}_s)(1-v)\Delta + 2^{-1}A(\bar{h}_s)f(A(\bar{h}_s)^{-1}k_{2,t}). \tag{4.17}$$

We show that for all integers $t \geq s$ relation (4.6) holds. By (4.9) for all integers $t \geq s$,

$$A(\bar{h}_s)^{-1}k_{2,t+1} \geq vA(\bar{h}_s)^{-1}k_{2,t} + f(A(\bar{h}_s)^{-1}k_{2,t}). \tag{4.18}$$

We claim that for all integers $t \geq s$,

$$A(\bar{h}_s)^{-1}k_{2,t} \geq g^{t-s}(\Delta). \tag{4.19}$$

By (4.3) and (4.7), relation (4.19) holds with $t = s$.

Assume that an integer $t \geq s$ and that (4.19) holds. By (4.18) and (4.19),

$$A(\bar{h}_s)^{-1}k_{2,t+1} \geq vg^{t-s}(\Delta) + f(g^{t-s}(\Delta)) = g^{t+1-s}(\Delta).$$

Thus (4.19) holds for all integers $t \geq s$. Together with (4.17) and monotonicity of f this implies that for all integers $t \geq s$,

$$k_{t+1} - vk_t \geq 2^{-1}A(\bar{h}_s)f(g^{t-s}(\Delta)).$$

Lemma 4.1 is proved.

By Proposition 3.1 there exists a natural number $\tau_0 > 3$ such that

$$g^{\tau_0 - 1}(\Delta) > x^*/2. \tag{4.20}$$

Lemma 4.2. *Let an integer* $s \geq 0$,

$$h_s \geq \bar{h}_s \geq \bar{h} \tag{4.21}$$

and

$$k_s \geq A(\bar{h}_s)\Delta. \tag{4.22}$$

Then there exists a a trajectory $(\{k_t\}_{t=s}^{s+\tau_0}, \{h_t\}_{t=s}^{s+\tau_0}, \{c_t\}_{t=s}^{s+\tau_0-1})$ *such that*

$$c_t = 2^{-1}A(\bar{h}_s)(f(\Delta) - (1-v)\Delta),\ t = s, \ldots, s+\tau_0-1,$$

$$h_t = h_s,\ \ t = s, \ldots, s+\tau_0-1,$$

$$h_{s+\tau_0} = h_s + 4^{-1}A(\bar{h}_s)f(x^*/2),$$

$$k_t \geq 2^{-1}A(\bar{h}_s)f(\Delta),\ t = s+1, \ldots, s+\tau_0-1,$$

$$k_{s+\tau_0} \geq 4^{-1}A(\bar{h}_s)f(x^*/2).$$

Proof. By Lemma 4.1 and (4.20) there exists a a trajectory

$$(\{\tilde{k}_t\}_{t=s}^{s+\tau_0}, \{\tilde{h}_t\}_{t=s}^{s+\tau_0}, \{\tilde{c}_t\}_{t=s}^{s+\tau_0-1})$$

such that

$$\tilde{k}_s = k_s, \tag{4.23}$$

$$\tilde{h}_t = h_s,\ t = s, \dots, s+\tau_0, \tag{4.24}$$

$$\tilde{c}_t = 2^{-1}A(\bar{h}_s)(f(\Delta) - (1-v)\Delta),\ t = s, \dots, s+\tau_0-1, \tag{4.25}$$

$$\tilde{k}_t \geq 2^{-1}A(\bar{h}_s)f(\Delta),\ t = s+1, \dots, s+\tau_0, \tag{4.26}$$

$$\tilde{k}_{s+\tau_0} - v\tilde{k}_{s+\tau_0-1} \geq 2^{-1}A(\bar{h}_s)f(g^{\tau_0-1}(\Delta)) \geq 2^{-1}A(\bar{h}_s)f(x^*/2). \tag{4.27}$$

In view of (4.24),

$$\tilde{k}_{s+\tau_0} - v\tilde{k}_{s+\tau_0-1} + \tilde{c}_{s+\tau_0-1} \leq A(\tilde{h}_{s+\tau_0-1})f(\tilde{k}_{s+\tau_0-1}A(h_s)^{-1}). \tag{4.28}$$

Set

$$k_t = \tilde{k}_t,\ t = s, \dots, s+\tau_0-1,\ c_t = \tilde{c}_t,\ t = s, \dots, s+\tau_0-1, \tag{4.29}$$

$$h_t = \tilde{h}_t = h_s,\ t = s, \dots, s+\tau_0-1, \tag{4.30}$$

$$k_{s+\tau_0} = vk_{s+\tau_0-1} + 4^{-1}A(\bar{h}_s)f(x^*/2), \tag{4.31}$$

$$h_{s+\tau_0} = h_s + 4^{-1}A(\bar{h}_s)f(x^*/2). \tag{4.32}$$

Let us show that $(\{k_t\}_{t=s}^{s+\tau_0}, \{h_t\}_{t=s}^{s+\tau_0}, \{c_t\}_{t=s}^{s+\tau_0-1})$ is a trajectory.

In order to meet this goal it is sufficient to show that

$$k_{s+\tau_0} - vk_{s+\tau_0-1} + c_{s+\tau_0-1} + h_{s+\tau_0} - h_{s+\tau_0-1}$$
$$\leq A(h_{s+\tau_0-1})f(k_{s+\tau_0-1}A(h_{s+\tau_0-1})^{-1}). \tag{4.33}$$

By (4.27)-(4.32),

$$\begin{aligned} &k_{s+\tau_0} - vk_{s+\tau_0-1} + c_{s+\tau_0-1} + h_{s+\tau_0} - h_{s+\tau_0-1} \\ &= 2^{-1}A(\bar{h}_s)f(x^*/2) + \tilde{c}_{s+\tau_0-1} \\ &\leq \tilde{k}_{s+\tau_0} - v\tilde{k}_{s+\tau_0-1} + \tilde{c}_{s+\tau_0-1} \\ &\leq A(\tilde{h}_{s+\tau_0-1})f(\tilde{k}_{s+\tau_0-1}A(\tilde{h}_{s+\tau_0-1})^{-1}) \\ &= A(h_{s+\tau_0-1})f(k_{s+\tau_0-1}A(h_{s+\tau_0-1})^{-1}). \end{aligned}$$

Thus (4.33) holds. This completes the proof of Lemma 4.2.

Choose an integer $q > 1$ such that

$$4^{-1}qA(\bar{h})f(x^*/2) \geq 1. \tag{4.34}$$

Lemma 4.3. *Let an integer* $p \geq 0$,

$$h_p \geq \bar{h}_p \geq \bar{h},$$

$$k_p \geq A(\bar{h}_p)\Delta. \tag{4.35}$$

Then there exists a a trajectory $(\{k_t\}_{t=p}^{p+q\tau_0}, \{h_t\}_{t=p}^{p+q\tau_0}, \{c_t\}_{t=p}^{p+q\tau_0-1})$ *such that*

$$c_t = 2^{-1}A(\bar{h}_p)(f(\Delta) - (1-v)\Delta),\ t = p, \dots, p + q\tau_0 - 1, \tag{4.36}$$

$$h_{p+q\tau_0} \geq h_p + 1, \tag{4.37}$$

$$k_{p+j\tau_0} \geq 4^{-1}A(\bar{h}_p)f(x^*/2),\ j = 1, \dots, q, \tag{4.38}$$

$$k_t \geq 2^{-1}A(\bar{h}_p)f(\Delta),\ t = p+1, \dots, p + q\tau_0. \tag{4.39}$$

Proof. We apply Lemma 4.2 with $s = p$ and obtain a trajectory

$$(\{k_t\}_{t=p}^{p+\tau_0}, \{h_t\}_{t=p}^{p+\tau_0}, \{c_t\}_{t=p}^{p+\tau_0-1})$$

such that

$$c_t = 2^{-1}A(\bar{h}_p)(f(\Delta) - (1-v)\Delta) \tag{4.40}$$

for all $t = p, \dots, p + \tau_0 - 1$,

$$h_{p+\tau_0} = h_p + 4^{-1}A(\bar{h}_p)f(x^*/2), \tag{4.41}$$

$$k_t \geq 2^{-1}A(\bar{h}_p)f(\Delta),\ t = p+1, \dots, p + \tau_0 - 1, \tag{4.42}$$

$$k_{p+\tau_0} \geq 4^{-1}A(\bar{h}_p)f(x^*/2). \tag{4.43}$$

Assume now that a natural number $j < q$ and we defined a a trajectory

$$(\{k_t\}_{t=p}^{p+j\tau_0}, \{h_t\}_{t=p}^{p+j\tau_0}, \{c_t\}_{t=p}^{p+j\tau_0-1})$$

such that (4.40) holds for $t = p, \dots, p + j\tau_0 - 1$, (4.42) holds for all $t = p + 1 \dots, p + j\tau_0 - 1$,

$$h_{p+j\tau_0} = h_p + 4^{-1}jA(\bar{h}_p)f(x^*/2), \tag{4.44}$$

$$k_{p+j\tau_0} \geq 4^{-1}A(\bar{h}_p)f(x^*/2). \tag{4.45}$$

(In view of (4.40)-(4.43) this assumption holds with $j = 1$.) Set

$$\bar{h}_{p+j\tau_0} = \bar{h}_p. \tag{4.46}$$

By (4.35) and (4.44)

$$h_{p+j\tau_0} \geq \bar{h}_{p+j\tau_0} \geq \bar{h}. \tag{4.47}$$

In view of (4.2), (4.45) and (4.46),

$$k_{p+j\tau_0} \geq A(\bar{h}_{p+j\tau_0})\Delta. \tag{4.48}$$

Using (4.46)-(4.48) we apply Lemma 4.2 with $s = p + j\tau_0$ and obtain a trajectory $(\{k_t\}_{t=p+j\tau_0}^{p+(j+1)\tau_0}, \{h_t\}_{t=p+j\tau_0}^{p+(j+1)\tau_0}, \{c_t\}_{t=p+j\tau_0}^{p+(j+1)\tau_0-1})$ such that (4.40) holds for all $t = p + j\tau_0, \dots, p + (j+1)\tau_0 - 1$, (4.42) holds for all $t = p + j\tau_0, \dots, p + (j+1)\tau_0 - 1$,

$$k_{p+(j+1)\tau_0} \geq 4^{-1}A(\bar{h}_p)f(x^*/2),$$

$$h_{p+(j+1)\tau_0} = h_{p+j\tau_0} + 4^{-1}A(\bar{h}_p)f(x^*/2) = h_p + 4^{-1}(j+1)A(\bar{h}_p)f(x^*/2).$$

(The last equality follows from (4.44).) Clearly the assumption made for j also holds for $j+1$ and by induction we construct a trajectory

$$(\{k_t\}_{t=p}^{p+q\tau_0}, \{h_t\}_{t=p}^{p+q\tau_0}, \{c_t\}_{t=p}^{p+q\tau_0-1})$$

such that (4.36), (4.38) and (4.39) hold and

$$h_{p+q\tau_0} = h_p + 4^{-1}qA(\bar{h}_p)f(x^*/2) \geq h_p + 1.$$

(The last inequality follows from (4.34).) Lemma 4.3 is proved.

Lemma 4.4. *Suppose that* $\Lambda_0 > 1$,

$$A(t+1)A(t)^{-1} \leq \Lambda_0 \text{ for all } t \geq \bar{h}, \tag{4.49}$$

$$0 < \Delta < (4\Lambda_0)^{-1}f(x^*/2). \tag{4.50}$$

Let an integer $p \geq 0$,

$$h_p \geq \bar{h}_p \geq \bar{h},\ k_p \geq A(\bar{h}_p)\Delta \tag{4.51}$$

and let a trajectory $(\{k_t\}_{t=p}^{p+q\tau_0}, \{h_t\}_{t=p}^{p+q\tau_0}, \{c_t\}_{t=p}^{p+q\tau_0-1})$ *be as guaranteed by Lemma 4.3. Then*

$$k_{p+q\tau_0} \geq A(\bar{h}_p+1)\Delta.$$

Proof. By Lemma 4.3, (4.38), (4.49), (4.50) and (4.51),

$$k_{p+q\tau_0} \geq 4^{-1}A(\bar{h}_p)f(x^*/2)$$

$$= A(\bar{h}_p+1)\Delta(f(x^*/2)\Delta^{-1})4^{-1}A(\bar{h}_p)A(\bar{h}_p+1)^{-1}$$

$$\geq A(\bar{h}_p+1)\Delta(4^{-1}f(x^*/2)\Delta^{-1}\Lambda_0^{-1}) \geq A(\bar{h}_p+1)\Delta.$$

Lemma 4.4 is proved.

Lemma 4.5. *Assume that for an integer* $p \geq 0$,

$$h_p \geq \bar{h}_p \geq \bar{h}, \tag{4.52}$$

$$k_p \geq A(\bar{h}_p)\Delta. \tag{4.53}$$

Then there exist an integer $\tau \geq p + q\tau_0$ *and a trajectory* $(\{k_t\}_{t=p}^{\tau}, \{h_t\}_{t=p}^{\tau}, \{c_t\}_{t=p}^{\tau-1})$ *such that*

$$h_\tau \geq h_p + 1,\ k_\tau \geq A(h_\tau)\Delta,$$

for all $t = p, \dots, \tau-1$,

$$c_t \geq 2^{-1}A(\bar{h}_p)\min\{f(\Delta) - (1-v)\Delta,\ f(f(\Delta)) - (1-v)f(\Delta)\}$$

and for all $t = p+1, \dots, \tau$,

$$k_t \geq 2^{-1}A(\bar{h}_p)\min\{\Delta,\ f(\Delta)\}.$$

Proof. By Lemma 4.3, (4.52) and (4.53), there exists a a trajectory

$$(\{k_t\}_{t=p}^{p+q\tau_0}, \{h_t\}_{t=p}^{p+q\tau_0}, \{c_t\}_{t=p}^{p+q\tau_0-1})$$

such that

$$c_t = 2^{-1}A(\bar{h}_p)(f(\Delta) - (1-v)\Delta),\ t = p, \dots, p + q\tau_0 - 1, \tag{4.54}$$

$$h_{p+q\tau_0} \ge h_p + 1, \tag{4.55}$$

$$k_{p+q\tau_0} \ge 4^{-1}A(\bar{h}_p)f(x^*/2), \tag{4.56}$$

$$k_t \ge 2^{-1}A(\bar{h}_p)f(\Delta),\ t = p+1, \dots, p + q\tau_0. \tag{4.57}$$

Set

$$k_{1,p+q\tau_0} = k_{p+q\tau_0},\ k_{2,p+q\tau_0} = k_{p+q\tau_0} \tag{4.58}$$

and for any integer $t \ge p + q\tau_0$ set

$$h_t = h_{p+q\tau_0}, \tag{4.59}$$

$$c_t = 2^{-1}A(\bar{h}_p)(f(f(\Delta)) - (1-v)f(\Delta)), \tag{4.60}$$

$$k_{1,t+1} = vk_{1,t} + A(h_{p+q\tau_0})f(k_{1,t}A(h_{p+q\tau_0})^{-1}) - 2c_t, \tag{4.61}$$

$$k_{2,t+1} = vk_{2,t} + A(h_{p+q\tau_0})f(k_{2,t}A(h_{p+q\tau_0})^{-1}), \tag{4.62}$$

$$k_{t+1} = vk_t + A(h_{p+q\tau_0})f(k_tA(h_{p+q\tau_0})^{-1}) - c_t. \tag{4.63}$$

By (4.55), (4.52), concavity of f and monotonicity of A for all $x \ge 0$,

$$A(h_{p+q\tau_0})f(xA(h_{p+q\tau_0})^{-1}) = A(h_{p+q\tau_0})f(xA(\bar{h}_p)^{-1}(A(\bar{h}_p)A(h_{p+q\tau_0})^{-1}))$$
$$\ge A(\bar{h}_p)f(xA(\bar{h}_p)^{-1}). \tag{4.64}$$

By (4.64), (4.61) and (4.62), for all integers $t \ge p + q\tau_0$,

$$k_{1,t+1} \ge vk_{1,t} + A(\bar{h}_p)f(k_{1,t}A(\bar{h}_p)^{-1}) - 2c_t, \tag{4.65}$$

$$k_{2,t+1} \ge vk_{2,t} + A(\bar{h}_p)f(k_{2,t}A(\bar{h}_p)^{-1}). \tag{4.66}$$

We will show that for all integers $t \ge p + q\tau_0$,

$$k_{1,t} \ge A(\bar{h}_p)f(\Delta). \tag{4.67}$$

By (4.56) and (4.2), relation (4.67) holds with $t = p + q\tau_0$. Assume that an integer $t \ge p + q\tau_0$ and (4.67) holds. By (4.65), (4.67), (4.54), (4.60) and monotonicity of f,

$$k_{1,t+1} \ge vA(\bar{h}_p)f(\Delta) + A(\bar{h}_p)f(f(\Delta)) - A(\bar{h}_p)(f(f(\Delta)) - (1-v)f(\Delta))$$
$$= A(\bar{h}_p)f(\Delta).$$

Thus (4.67) holds for all integers $t \ge p + q\tau_0$.

By (4.65), (4.60) and (4.67) for all integers $t \ge p + q\tau_0$,

$$k_{1,t+1} - vk_{1,t} \ge A(\bar{h}_p)f(f(\Delta)) - A(\bar{h}_p)(f(f(\Delta)) - (1-v)f(\Delta))$$
$$= A(\bar{h}_p)(1-v)f(\Delta). \tag{4.68}$$

In view of (4.61), for all integers $t \ge p + q\tau_0$,

$$k_{1,t+1} - vk_{1,t} + 2c_t \le A(h_{p+q\tau_0})f(k_{1,t}A(h_{p+q\tau_0})^{-1}). \tag{4.69}$$

We show that for all integers $t \geq p + q\tau_0$,

$$k_t \geq 2^{-1}(k_{1,t} + k_{2,t}). \tag{4.70}$$

By (4.58) inequality (4.70) holds for $t = p + q\tau_0$.

Assume that an integer $t \geq p+q\tau_0$ and that (4.70) holds. By (4.61)-(4.63), (4.70) and monotonicity and concavity of f,

$$k_{t+1} - 2^{-1}(k_{1,t+1} + k_{2,t+1})$$

$$= vk_t + A(h_{p+q\tau_0})f(k_tA(h_{p+q\tau_0})^{-1}) - c_t - 2^{-1}(vk_{1,t} + vk_{2,t})$$

$$-2^{-1}A(h_{p+q\tau_0})(f(k_{1,t}A(h_{p+q\tau_0})^{-1}) + f(k_{2,t}A(h_{p+q\tau_0})^{-1}) + c_t$$

$$\geq A(h_{p+q\tau_0})[f(k_tA(h_{p+q\tau_0})^{-1})$$

$$-2^{-1}f(k_{1,t}A(h_{p+q\tau_0})^{-1}) - 2^{-1}f(k_{2,t}A(h_{p+q\tau_0})^{-1})] \geq 0.$$

Thus (4.70) holds for all integers $t \geq p + q\tau_0$.

By (4.63), (4.70), (4.60), (4.64), (4.67) and concavity of f for all integers $t \geq p + q\tau_0$,

$$k_{t+1} - vk_t \geq A(h_{p+q\tau_0})f(A(h_{p+q\tau_0})^{-1}k_t) - c_t$$

$$\geq 2^{-1}A(h_{p+q\tau_0})f(A(h_{p+q\tau_0})^{-1}k_{1,t}) + 2^{-1}A(h_{p+q\tau_0})f(A(h_{p+q\tau_0})^{-1}k_{2,t}) - c_t$$

$$\geq 2^{-1}A(\bar{h}_p)f(A(\bar{h}_p)^{-1}k_{1,t}) + 2^{-1}A(\bar{h}_p)f(A(\bar{h}_p)^{-1}k_{2,t}) - c_t$$

$$\geq 2^{-1}A(\bar{h}_p)f(f(\Delta)) - 2^{-1}A(\bar{h}_p)(f(f(\Delta)) - (1-v)f(\Delta))$$

$$+2^{-1}A(\bar{h}_p)f(A(\bar{h}_p)^{-1}k_{2,t})$$

$$\geq 2^{-1}(1-v)f(\Delta) + 2^{-1}A(\bar{h}_p)f(A(\bar{h}_p)^{-1}k_{2,t}). \tag{4.71}$$

By (4.62) for all integers $t \geq p + q\tau_0$,

$$A(h_{p+q\tau_0})^{-1}k_{2,t+1} = vA(h_{p+q\tau_0})^{-1}k_{2,t} + f(A(h_{p+q\tau_0})^{-1}k_{2,t}). \tag{4.72}$$

By (4.56), (4.58), (4.72) and Proposition 3.1, there is an integer $\tau > p + q\tau_0$ such that

$$k_{2,\tau} > 2^{-1}x^*A(h_{p+q\tau_0}). \tag{4.73}$$

Now it is not difficult see that in view of (4.59), (4.60), (4.63) and (4.71),

$$(\{k_t\}_{t=p}^{\tau}, \{h_t\}_{t=p}^{\tau}, \{c_t\}_{t=p}^{\tau-1})$$

is a trajectory. By (4.59), (4.70) and (4.73),

$$k_\tau > 2^{-1}k_{2,\tau} \geq 4^{-1}x^*A(h_{p+q\tau_0}) \geq \Delta A(h_\tau).$$

Lemma 4.5 is proved.

5. Proof of Theorem 3.5

We use the notation, definitions and assumptions introduced in Sections 2 and 3.

Let

$$\bar{h}_0 = \bar{h}. \tag{5.1}$$

By (5.1) and Lemmas 4.3 and 4.4 applied with $p = 0$ there exists a trajectory $(\{k_t\}_{t=0}^{q\tau_0}, \{h_t\}_{t=0}^{q\tau_0}, \{c_t\}_{t=0}^{q\tau_0-1})$ such that

$$c_t = 2^{-1}A(\bar{h})(f(\Delta) - (1-v)\Delta),\ t = 0, \dots, q\tau_0 - 1, \tag{5.2}$$

$$h_{q\tau_0} \geq h_0 + 1, \tag{5.3}$$

$$k_{q\tau_0} \geq A(\bar{h}+1)\Delta, \tag{5.4}$$

$$k_t \geq 2^{-1}A(\bar{h})f(\Delta),\ t = 1, \dots, q\tau_0. \tag{5.5}$$

Assume that s is a natural number and we constructed a trajectory

$$(\{k_t\}_{t=0}^{sq\tau_0}, \{h_t\}_{t=0}^{sq\tau_0}, \{c_t\}_{t=0}^{sq\tau_0-1})$$

such that for all integers $j = 1, \dots, s$,

$$k_{jq\tau_0} \geq A(\bar{h}+j)\Delta,\ h_{jq\tau_0} \geq h_0 + j, \tag{5.6}$$

$$c_t = 2^{-1}A(\bar{h}+j-1)(f(\Delta) - (1-v)\Delta),\ t = (j-1)q\tau_0, \dots, jq\tau_0 - 1, \tag{5.7}$$

$$k_t \geq 2^{-1}A(\bar{h}+j-1)f(\Delta),\ t = (j-1)q\tau_0 + 1, \dots, jq\tau_0. \tag{5.8}$$

(Note that in view of (5.2), (5.3), (5.4) and (5.5) our assumptions hold for $j = 1$.)

Let

$$p = sq\tau_0,\ \bar{h}_{sq\tau_0} = \bar{h}_p = \bar{h} + s. \tag{5.9}$$

In view of (5.6), (5.9) and the relation $h_0 \geq \bar{h}$,

$$h_p = h_{sq\tau_0} \geq h_0 + s \geq \bar{h}_{sq\tau_0} = \bar{h}_p. \tag{5.10}$$

By (5.6) and (5.9),

$$k_p = k_{sq\tau_0} \geq A(\bar{h}+s)\Delta = A(\bar{h}_p)\Delta. \tag{5.11}$$

Using (5.9), (5.10) and (5.11) and applying Lemmas 4.3 and 4.4 we obtain that there exists a a trajectory $(\{k_t\}_{t=sq\tau_0}^{(s+1)q\tau_0}, \{h_t\}_{t=sq\tau_0}^{(s+1)q\tau_0}, \{c_t\}_{t=sq\tau_0}^{(s+1)q\tau_0-1})$ such that

$$h_{(s+1)q\tau_0} \geq h_{sq\tau_0} + 1 \geq h_0 + s + 1,$$

$$k_{(s+1)q\tau_0} \geq A(\bar{h}_p + 1)\Delta = A(\bar{h} + s + 1)\Delta,$$

$$c_t = 2^{-1}A(\bar{h}+s)(f(\Delta) - (1-v)\Delta),\ t = sq\tau_0, \dots, (s+1)q\tau_0 - 1,$$

$$k_t \geq 2^{-1}A(\bar{h}+s)f(\Delta),\ t = sq\tau_0 + 1, \dots, (s+1)q\tau_0.$$

It follows from the relations above that the assumption made for s also holds for $s+1$. Therefore by induction we constructed a trajectory $(\{k_t\}_{t=0}^{\infty}, \{h_t\}_{t=0}^{\infty}, \{c_t\}_{t=0}^{\infty})$ such that for all natural numbers j (5.6) and (5.8) hold.

Assume that an integer $t \geq 0$. There is a natural number j such that

$$(j-1)q\tau_0 \leq t < jq\tau_0.$$

We have

$$j-1=[tq^{-1}\tau_0^{-1}],\ j=[tq^{-1}\tau_0^{-1}]+1. \tag{5.12}$$

By (5.6) and (5.12),

$$h_t \geq h_{(j-1)q\tau_0} \geq h_0+(j-1)=h_0+[tq^{-1}\tau_0^{-1}]. \tag{5.13}$$

It follows from (5.8) and (5.12) that

$$k_{t+1} \geq 2^{-1}A(\bar{h}+j-1)f(\Delta)=2^{-1}A(\bar{h}+[tq^{-1}\tau_0^{-1}])f(\Delta).$$

By (5.7) and (5.12),

$$c_t=2^{-1}A(\bar{h}+j-1)(f(\Delta)-(1-v)\Delta)=2^{-1}A(\bar{h}+[tq^{-1}\tau_0^{-1}])(f(\Delta)-(1-v)\Delta).$$

Theorem 3.5 is proved.

6. Proof of Theorem 3.6

We use the notation, definitions and assumptions of Sections 2 and 3. Set

$$\Delta_0=\min\{\Delta,\ f(\Delta)-(1-v)\Delta,\ f(f(\Delta))-(1-v)f(\Delta)\}.$$

Let

$$\bar{h}_0=h_0,\ T_0=0. \tag{6.1}$$

By Lemma 4.5 with $p=0$ there exist an integer $T_1>T_0$ and a trajectory

$$(\{k_t\}_{t=0}^{T_1},\{h_t\}_{t=0}^{T_1},\{c_t\}_{t=0}^{T_1-1})$$

such that

$$h_{T_1} \geq h_{T_0}+1=h_0+1,\ k_{T_1} \geq A(h_{T_1})\Delta, \tag{6.2}$$

for all $t=0,\dots,T(1)-1$,

$$c_t \geq 2^{-1}A(h_{T_0})\Delta_0, \tag{6.3}$$

for all $t=T_0+1,\dots,T_1$,

$$k_t \geq 2^{-1}A(h_{T_0})\Delta_0. \tag{6.4}$$

Assume that s is a natural number and we constructed a strictly increasing sequence of integers $\{T_j\}_{j=0}^{s}$ such that $T_0=0$ and a trajectory

$$(\{k_t\}_{t=0}^{T_s},\{h_t\}_{t=0}^{T_s},\{c_t\}_{t=0}^{T_s-1})$$

such that for all integers $j=0,\dots,s$,

$$h_{T_j} \geq h_0+j,\ k_{T_j} \geq A(h_{T_j})\Delta, \tag{6.5}$$

for all integers $j=0,\dots,s-1$,

$$c_t \geq 2^{-1}A(h_{T_j})\Delta_0,\ t=T_j,\dots,T_{j+1}-1 \tag{6.6}$$

and

$$k_t \geq 2^{-1}A(h_{T_j})\Delta_0,\ t=T_j+1,\dots,T_{j+1}. \tag{6.7}$$

(By (6.2) and (6.3) our assumption hold for $s = 1$.) Using (6.5) and applying Lemma 4.5 with $p = T_s$, $\bar{h}_p = h_{T_s}$ we obtain that there exist an integer $T_{s+1} > T_s$ and a trajectory $(\{k_t\}_{t=T_s}^{T_{s+1}}, \{h_t\}_{t=T_s}^{T_{s+1}}, \{c_t\}_{t=T_s}^{T_{s+1}-1})$ such that

$$h_{T_{s+1}} \geq h_{T_s} + 1,\ k_{T_{s+1}} \geq A(h_{T_{s+1}})\Delta,$$

for all $t = T_s, \dots, T_{s+1} - 1$,

$$c_t \geq 2^{-1}A(h_{T_s})\Delta_0,$$

for all $t = T_s + 1, \dots, T_{s+1}$,

$$k_t \geq 2^{-1}A(h_{T_s})\Delta_0.$$

Now it is not difficult to see that the assumption made for s also holds for $s + 1$. Therefore by induction we constructed the trajectory and Theorem 3.6 is proved.

7. Summary

Endogenous growth models are typically built on a specific economic structure in the form of technology, preferences and a growth mechanism (e.g., knowledge or human capital that augment labor productivity, indefinite number of intermediate capital good varieties or consumer good varieties, creative destructive R&D processes), with the function specifications and ranges of parameter values so adjusted in order to obtain balanced (exponential) growth in the long run. This fine tuning often exhibits a knife-edge property, in that a small deviation in a parameter value results in the economy growing too fast, reaching infinity at a finite time, or too slow, eventually converging to a stationary steady state (see Solow 1994, 2000).

We study the conditions under which unbounded endogenous growth is feasible in the context of a general economic structure. The underlying technology is loosely specified (requiring it to satisfy very mild conditions) and the growth mechanism considered is that of labor-augmenting human capital that competes with ordinary capital and consumption for the available resources (as in Lucas, 1988). When unbounded endogenous growth is feasible we provide lower bounds on the long-run rate of growth in terms of properties of the labor-augmenting and production functions.

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