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NONLINEAR QUADRATIC VOLTERRA RIEMANN-LIOUVILLE INTEGRAL EQUATIONS OF FRACTIONAL ORDER

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Abstract. In this paper, we prove an existence result for a nonlinear quadratic Volterra integral equation of fractional order. Our technique is based on a fixed point theorem due to Dhage [12].

1. Introduction

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus and goes back to times when G.W. Leibniz and I. Newton invented differential calculus. The theory of differential and integral equations of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. We can find numerous applications of differential and integral equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [13, 15, 16, 17, 20, 21]. There has been a significant development in ordinary and partial fractional and integral differential equations in recent years; see the monographs of Abbas *et al.* [7], Kilbas *et al.* [18], Lakshmikantham *et al.* [19], Miller and Ross [22], Podlubny [23], Samko *et al.* [24], the papers of Abbas and Benchohra [3, 4], Abbas *et al.* [2, 5, 6], Benchohra *et*

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al. [9, 10, 11], Diethelm [13, 14], Mainardi [20], Vityuk and Golushkov [25], Zhang [26] and the references therein.

In [8], Abbas *et al.* proved an existence result for partial discontinuous hyperbolic differential equations of fractional order in Banach algebras under Lipschitz and Carathéodory conditions by using some fixed point theorems of Dhage for the product of two operators. Recently, in [1], Mohamed I. Abbas studied the existence of solutions in the space of real functions defined, continuous and bounded on an unbounded interval of the following nonlinear quadratic Volterra integral equation of fractional order

$$x(t) = [f(t, x(t))]\Big(q(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(t, s, x(s))}{(t-s)^{1-\alpha}} ds\Big);\ t \in [0, \infty),$$

where $\alpha \in (0, 1]$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt,\ \xi > 0.$$

This paper deals with the existence of solutions to the following nonlinear quadratic Volterra integral equation of Riemann-Liouville fractional order

$$u(x, y) = f(x, y, u(x, y))\Big[\mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}$$

$$\times \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(x, y, s, t, u(s, t)) dt ds\Big],$$

$$\text{if } (x, y) \in J := [0, a] \times [0, b], \tag{1}$$

where $a, b > 0,\ r_1, r_2 \in (0, \infty),\ f : J \times \mathbb{R} \to \mathbb{R},\ g : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ and $\mu : J \to \mathbb{R}$ are given continuous functions, where

$$\mathcal{D} = \{((x, y), (s, t)) \in J \times J : s \le x \text{ and } t \le y\}.$$

We employ a hybrid fixed point theorem of Dhage [12] for proving our existence result. Finally, an example illustrating the main existence result is presented in the last Section.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $L^1(J)$ be denote the space of Lebesgue-integrable functions $u : J \to \mathbb{R}$ with the norm

$$\|u\|_1 = \int_0^a \int_0^b |u(x, y)| dy dx.$$

By $C(J)$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|u\|_\infty = \sup_{(x,y)\in J} |u(x, y)|.$$

Define a multiplication " $\cdot$ " by

$$(u \cdot v)(x, y) = u(x, y)v(x, y)$$

for each $(x, y) \in J$, then $C(J)$ is a Banach algebra with above norm and multiplication.

Definition 2.1 ([25])**.** Let $\theta = (0,0)$, $r_1,\ r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J)$, the expression

$$(I_\theta^r u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t)dtds,$$

is called the left-sided mixed Riemann-Liouville integral of order r.

In particular,

$$(I_\theta^\theta u)(x,y) = u(x,y),\ (I_\theta^\sigma u)(x,y) = \int_0^x \int_0^y u(s,t)dtds; \text{ for almost all } (x,y) \in J,$$

where $\sigma = (1,1)$.
For instance, $I_\theta^r u$ exists for all $r_1, r_2 > 0$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I_\theta^r u) \in C(J)$, moreover

$$(I_\theta^r u)(x,0) = (I_\theta^r u)(0,y) = 0;\ x \in [0,a],\ y \in [0,b].$$

Example 2.2. *Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0,\infty) \times (0, \infty)$, then*

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2},\ \textit{for almost all } (x,y) \in J.$$

We use the following fixed point theorem of Dhage for proving the existence of solutions for our equation.

Theorem 2.3 (Dhage [12])**.** *Let D be a closed-convex and bounded subset of the Banach algebra X and let $F, G : D \to X$ be two operators satisfying:*

(a) *A is Lipschitz with the Lipschitz constant λ,*
(b) *B is completely continuous,*
(c) *$Au\,Bz \in D$ for all $u, z \in D$, and*
(d) *$\lambda M < 1$ where $M = \|B(D)\| = \sup_{u \in D} \|B(u)\|$.*

Then the operator equation $AuBu = u$ has a solution and the set of all solutions is compact in D.

3. Existence of Solutions

In this section, we are concerned with the existence of solutions for the equation (1). The following hypotheses will be used in the sequel.

(A1) There exists a positive continuous function $\alpha : J \to \mathbb{R}$ such that

$$|f(x,y,u) - f(x,y,v)| \le \alpha(x,y)|u-v|, \text{ for all } (x,y) \in J, \quad \text{and} \quad u, v \in \mathbb{R}.$$

(A2) There exist a positive continuous function $\beta : J \to \mathbb{R}$ and a positive bounded continuous function $h : [0,\infty) \to \mathbb{R}$ with $h(0) = 0$ such that for all $((x,y),(s,t)) \in \mathcal{D}$, and $u, v \in \mathbb{R}$,

$$|g(x,y,s,t,u) - g(x,y,s,t,v)| \le \beta(x,y)h(|u-v|).$$

Set

$$K = \sup_{\eta>0} \frac{a^{r_1} b^{r_2} [g^* + \|\beta\|_\infty h(\eta)]}{\Gamma(1+r_1)\Gamma(1+r_2)},$$

where $g^* = \sup\{g(x,y,s,t,0) : ((x,y),(s,t)) \in \mathcal{D}\}$.

Theorem 3.1. *Assume that hypotheses (A1) and (A2) hold. If*

$$\|\alpha\|_\infty (K + \|\mu\|_\infty) < 1, \tag{2}$$

then the equation (1) *has at least one solution on J.*

Proof. Consider the closed ball

$$D := \{u \in C(J) : \|u\|_\infty \le \rho\},$$

where

$$\rho = \frac{f^*(K + \|\mu\|_\infty)}{1 - \|\alpha\|_\infty (K + \|\mu\|_\infty)} > 0,$$

and

$$f^* = \sup_{(x,y)\in J} \|f(x,y,0)\|.$$

Let us define two operators A and B on D by

$$Au(x,y) = f(x,y,u(x,y)); \quad (x,y) \in J, \tag{3}$$

$$Bu(x,y) = \mu(x,y)$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(x,y,s,t,u(s,t))dtds; \ (x,y) \in J. \tag{4}$$

Clearly A and B define the operators $A, \ B : D \to C(J)$. Now solving the equation (1) is equivalent to solving the operator equation

$$Au(x,y)\, Bu(x,y) = u(x,y); \quad (x,y) \in J. \tag{5}$$

We show that operators A and B satisfy all the assumptions of Theorem 2.3.
First we shall show that A is a Lipschitz. Let $u_1, u_2 \in D$. Then by (A1), for all $(x,y) \in J$, we get

$$\begin{aligned} |Au_1(x,y) - Au_2(x,y)| &= |f(x,y,u_1(x,y)) - f(x,y,u_2(x,y))| \\ &\le \alpha(x,y)|u_1(x,y) - u_2(x,y)|. \end{aligned}$$

Taking the maximum over (x,y), in the above inequality yields

$$\|Au_1 - Au_2\|_\infty \le \|\alpha\|_\infty \|u_1 - u_2\|_\infty,$$

and so A is a Lipschitz with a Lipschitz constant $\|\alpha\|_\infty$.

Next, we show that B is a continuous and compact operator on D. The proof will be given in several steps.

Step 1: B *is continuous.*
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in D. Then, for each $(x,y) \in J$, we have

$$|B(u_n)(x,y) - B(u)(x,y)|$$

$$\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y |x-s|^{r_1-1}|y-t|^{r_2-1}|g(x,y,s,t,u_n(s,t))$$
$$-g(x,y,s,t,u(s,t))|dtds$$
$$\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b |x-s|^{r_1-1}|y-t|^{r_2-1} \sup_{(x,y)\in J} \beta(x,y)h(\|u_n-u\|_\infty)dtds$$
$$\leq \frac{\|\beta\|_\infty h(\|u_n-u\|_\infty)}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b |x-s|^{r_1-1}|y-t|^{r_2-1}dtds$$
$$\leq \frac{a^{r_1}b^{r_2}\|\|\beta\|_\infty h(\|u_n-u\|_\infty)}{\Gamma(1+r_1)\Gamma(1+r_2)}.$$

Since h is a continuous function, we have

$$\|B(u_n)-B(u)\|_\infty \leq \frac{a^{r_1}b^{r_2}\|\beta\|_\infty h(\|u_n-u\|_\infty)}{\Gamma(1+r_1)\Gamma(1+r_2)} \to 0 \text{ as } n\to\infty.$$

Step 2: $B(D)$ *is bounded.*
Indeed, it is enough show that there exists a positive constant M^* such that, for each $u \in D$, we have $\|B(u)\| \leq M^*$. Let $u \in D$ be arbitrary, then for each $(x,y) \in J$, we have

$$\begin{aligned}|Bu(x,y)| &\leq |\mu(x,y)| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}|g(x,y,s,t,u(s,t))|dtds \\ &\leq \|\mu\|_\infty + \frac{a^{r_1}b^{r_2}[g^*+\|\beta\|_\infty h(\rho)]}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &\leq K + \|\mu\|_\infty := M^*.\end{aligned}$$

Step 3: $B(D)$ *is equicontinuous.*
Let $(x_1,y_1),(x_2,y_2) \in J$, $x_1 < x_2$, $y_1 < y_2$ and $u \in D$. Then

$$|B(u)(x_2,y_2) - B(u)(x_1,y_1)|$$
$$= |\mu(x_1,y_1) - \mu(x_2,y_2)|$$
$$+\Big|\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_0^{y_1} [(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} - (x_1-s)^{r_1-1}(y_1-t)^{r_2-1}]$$
$$\times g(x,y,s,t,u(s,t))dtds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}g(x,y,s,t,u(s,t))dtds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}g(x,y,s,t,u(s,t))dtds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_0^{y_1} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}g(x,y,s,t,u(s,t))dtds\Big|$$
$$\leq |\mu(x_1,y_1) - \mu(x_2,y_2)| + \frac{g^*+\|\beta\|_\infty h(\rho)}{\Gamma(r_1)\Gamma(r_2)}$$

$$\times \int_0^x \int_0^y [(x_1-s)^{r_1-1}(y_1-t)^{r_2-1} - (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}]dtds$$
$$+\frac{g^* + \|\beta\|_\infty h(\rho)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds$$
$$+\frac{g^* + \|\beta\|_\infty h(\rho)}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds$$
$$+\frac{g^* + \|\beta\|_\infty h(\rho)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_0^{y_1} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds$$
$$\leq |\mu(x_1,y_1) - \mu(x_2,y_2)|$$
$$+\frac{[g^* + \|\beta\|_\infty h(\rho)]a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}[2y_2^{r_2}(x_2-x_1)^{r_1} + 2x_2^{r_1}(y_2-y_1)^{r_2}$$
$$+x_1^{r_1}y_1^{r_2} - x_2^{r_1}y_2^{r_2} - 2(x_2-x_1)^{r_1}(y_2-y_1)^{r_2}].$$

As $x_1 \to x_2$, $y_1 \to y_2$ the right-hand side of the above inequality tends to zero.
As a consequence of steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that B is continuous and compact.

Next, we show that $Au\ Bu \in D$ for all $u \in D$. Let $u \in D$ be arbitrary, then for each $(x,y) \in J$,

$$\begin{aligned}
|Au(x,y)Bu(x,y)| &\leq (f^* + \rho\|\alpha\|_\infty)\Big[|\mu(x,y)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
&\quad \times \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}|g(x,y,s,t,u(s,t))|dtds\Big] \\
&\leq (f^* + \rho\|\alpha\|_\infty)\Big[\|\mu\|_\infty + \frac{a^{r_1}b^{r_2}[g^* + \|\beta\|_\infty h(\rho)]}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big] \\
&\leq (f^* + \rho\|\alpha\|_\infty)(\|\mu\|_\infty + K) \\
&= \frac{f^*(k + \|\mu\|_\infty)}{1 - \|\alpha\|_\infty(K + \|\mu\|_\infty)} \\
&= \rho.
\end{aligned}$$

Also, we have

$$M = \|B(D)\| \leq K + \|\mu\|_\infty,$$

and therefore, by (2), we get,

$$M\|\alpha\|_\infty \leq \|\alpha\|_\infty(K + \|\mu\|_\infty) < 1.$$

Now we apply Theorem 2.3 to conclude that the equation (1) has a solution on J and the set of all solutions is compact in D.

4. An Example

As an application of our results we consider the following quadratic Volterra integral equation of fractional order

$$u(x,y) = [xy^2 + xyu(x,y)]\Big[xye^{-(x^2+y^2)}$$

$$+\frac{1}{\Gamma(\frac{2}{3})\Gamma(\frac{3}{4})}\int_0^x\int_0^y (x-s)^{-\frac{1}{3}}(y-t)^{-\frac{1}{4}}\Big(xt+\frac{e^{-(x+y+s+t)}}{1+u^{\frac{2}{3}}(s,t)}\Big)dtds\Big];\; x,y\in[0,1], \tag{6}$$

where $r = (\frac{2}{3}, \frac{3}{4})$, $\mu : [0,1]\times[0,1] \to \mathbb{R}$, $f : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ and $g : \mathcal{D}\times\mathbb{R} \to \mathbb{R}$ defined by

$$\mu(x,y) = xye^{-(x^2+y^2)},$$

$$f(x,y,u) = xy^2 + xyu,$$

and

$$g(x,y,s,t,u) = xt + \frac{e^{-(x+y+s+t)}}{1+u^{\frac{2}{3}}(s,t)}.$$

Here the set $\mathcal{D}$ is defined by

$$\mathcal{D} = \{((x,y),(s,t)) \in [0,1]\times[0,1]\times[0,1]\times[0,1] : s \le x \text{ and } t \le y\}.$$

Then we can easily check that the assumptions of Theorem 3.1 are satisfied. In fact, we have that the function f is continuous and satisfies assumption (A1), where $\alpha(x,y) = xy$, then $\|\alpha\|_\infty = 1$ and $f^* = 1$. Next, let us notice that the function g satisfies assumption (A2), where $\beta(x,y) = e^{-(x+y)}$, $h(\rho) = \frac{1}{1+\rho^{\frac{2}{3}}}$ and $g^* = 1$. Also, condition (2) is satisfied. Indeed,

$$\|\mu\|_\infty \le e^{-2},$$

and

$$\begin{aligned} K &= \sup_{\eta>0}\frac{a^{r_1}b^{r_2}[g^* + \|\beta\|_\infty h(\eta)]}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &= \frac{1}{\Gamma(\frac{5}{3})\Gamma(\frac{7}{4})}. \end{aligned}$$

Thus

$$\begin{aligned} \|\alpha\|_\infty(K + \|\mu\|_\infty) &\le \frac{1}{\Gamma(\frac{5}{3})\Gamma(\frac{7}{4})} + e^{-2} \\ &< 0.64 + 0.13 \\ &= 0.77 \\ &< 1. \end{aligned}$$

Hence by Theorem 3.1, the equation (6) has a solution defined on $[0,1]\times[0,1]$.

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