DUALITY IN MULTIOBJECTIVE FRACTIONAL VARIATIONAL CONTROL PROBLEMS WITH GENERALIZED $F$-INVEXITY

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Abstract. In this paper, a class of multiobjective fractional variational control problems with generalized $F$-invexity are introduced and studied. Some relations of the efficient solutions between the multiobjective fractional variational control problems and its Wolfe duality problems are established under some suitable conditions.

1. Introduction

Convexity plays a vital role in optimization theory and some related fields. However, the classic convexity could not satisfy the requirement for solving many practical problems. Thus, many kinds of generalized convexity sets and generalized convexity functions have been introduced and studied in the past decades. The first type of generalized convexity was considered by Finetti [4] who first introduced the quasiconvex functions. Mangasarian [10] considered the family of pseudoconvex functions. Then, Ponstein [13] and Katzner [8] extended them to strictly quasiconvex functions and strictly pseudoconvex functions (see Giorgi [5]).

On the other hand, Egudo [3] used the concept of efficiency (Pareto optimum) to formulate duality for multiobjective nonlinear programs. Hanson [6] extended the Wolfe-duality results of mathematical programming to a class of functions subsequently called invex function. Jo [7] considered a multiobjective fractional programming problem involving vector-valued objective n-set functions. Later, Chen [2] studied the Optimality and duality for the multiobjective fractional programming with the generalized $(F, \rho)$-convexity. More recently, a number of optimality criteria and duality relations for various classes of generalized fractional programming problems have appeared in the related literature. Kim and Kim [9] considered

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the Wolfe type dual for a nondifferentiable multiobjective variational problems under \((F, \rho)\)-convexity assumptions. Nahak and Nanda [12] extended the concept of \(V\)-invexity to continuous case and used it to prove sufficient optimality and duality results for a class of variational control problems. Arana [1] proved that a KT-invex control problem is characterized in order that a Kuhn-Tucker point is an optimal solution and generalized optimality results of known mathematical programming problems.

Motivated and inspired by the research works mentioned above, in this paper, we introduce and study a class of multiobjective fractional variational control problems with generalized \(F\)-invexity. We establish some relations of the efficient solutions between the multiobjective fractional variational control problems and its Wolfe duality problems under some suitable conditions.

2. Preliminaries

Let \(K = \{1, 2, \cdots, k\}\), the interval \(I = [a, b]\) be a real interval and the functions \(\psi : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}\) be continuously differentiable. In order to consider \(\psi(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\), where \(x(t) : I \to \mathbb{R}^n, u(t) : I \to \mathbb{R}^l\) with derivative \(\dot{x}(t)\) and \(\dot{u}(t)\), \(t\) is the independent variable, \(x(t)\) is the state variable and \(u(t)\) is the control variable, denote the partial derivative of the scalar function \(\psi(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\), respectively, by \(\psi_t, \psi_x, \psi_{\dot{x}}, \psi_u, \psi_{\dot{u}}\). For example,

\[
\psi_x = \left(\frac{\partial \psi}{\partial x^1}, \cdots, \frac{\partial \psi}{\partial x^n}\right), \quad \psi_{\dot{x}}(t) = \left(\frac{\partial \psi}{\partial \dot{x}_1}, \cdots, \frac{\partial \psi}{\partial \dot{x}_n}\right).
\]

The partial derivatives of other functions used will be written similarly. The partial derivatives of other functions used will be written similarly. Let \(S(I, \mathbb{R}^n)\) denote the space of piecewise smooth functions \(x : I \to \mathbb{R}^n\) with norm

\[
\|x\| = \|x\|_{\infty} + \|Dx\|_{\infty},
\]

where the differentiation operator is given by

\[
u = Dx \longleftrightarrow x(t) = \alpha + \int u(s)ds,
\]

in which \(\alpha\) is a given boundary value. Therefore, \(D = \frac{d}{dt}\) except at discontinuities. Let \(X_0 \subset S(I, \mathbb{R}^n)\) and \(U_0 \subset S(I, \mathbb{R}^l)\) be two nonempty subsets. Let \(\tilde{f} : X_0 \times U_0 \to \mathbb{R}\) defined by

\[
\tilde{f}^i(t(x), u(t)) = f^i(t, x(t), \dot{x}(t), u(t), \dot{u}(t))
\]

be Frechet differentiable.

Now we consider the following multiobjective fractional control problem:

\[
(MFP) \quad \min \left(\frac{\int_a^b p^1(t, x, \dot{x}, u, \dot{u})dt + (x^T B_1 x)^{\frac{1}{2}} dt}{\int_a^b q^1(t, x, \dot{x}, u, \dot{u})dt}, \ldots, \frac{\int_a^b p^k(t, x, \dot{x}, u, \dot{u}) + (x^T B_k x)^{\frac{1}{2}} dt}{\int_a^b q^k(t, x, \dot{x}, u, \dot{u})dt}\right)
\]

subject to \(x(a) = \alpha, \quad x(b) = \beta, \quad g(t, x, \dot{x}, u, \dot{u}) \leq 0, \quad x \in C(I, \mathbb{R}^n),\)

where \(p^i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}, q^i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}\) and \(g : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}^m\) are assumed to be continuously differentiable.
functions with \( i \in K \), and for each \( t \in I \) and \( i \in K \), \( B_i \) is an \( n \times n \) positive semidefinite (symmetric) matrix.

**Definition 2.1.** A point \( (x^*, u^*) \) is said to be an efficient solution for \((MFP)\) if there exists no other \((x, u)\) such that

\[
\frac{\int_a^b p^i(t, x, \dot{x}, u, \dot{u}) + (x^T B_i x)^{\frac{1}{2}} dt}{\int_a^b q^i(t, x, \dot{x}, u, \dot{u}) dt} \leq \frac{\int_a^b p^j(t, x^*, \dot{x}^*, u^*, \dot{u}^*) + (x^{*T} B_j x^*)^{\frac{1}{2}} dt}{\int_a^b q^j(t, x^*, \dot{x}^*, u^*, \dot{u}^*) dt}
\]

for all \( i \in K \) and

\[
\frac{\int_a^b p^i(t, x, \dot{x}, u, \dot{u}) + (x^T B_i x)^{\frac{1}{2}} dt}{\int_a^b q^i(t, x, \dot{x}, u, \dot{u}) dt} < \frac{\int_a^b p^j(t, x^*, \dot{x}^*, u^*, \dot{u}^*) + (x^{*T} B_j x^*)^{\frac{1}{2}} dt}{\int_a^b q^j(t, x^*, \dot{x}^*, u^*, \dot{u}^*) dt}
\]

for some \( j \in K \).

**Definition 2.2.** A functional \( F : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be sublinear if, for any \((x, u), (x^*, u^*) \in X_0 \times U_0, \)

\[
F(t, x, \dot{x}, x^*, \dot{x}^*, u, \dot{u}, u^*, \dot{u}^*; \alpha_1 + \alpha_2) \\
\leq F(t, x, \dot{x}, x^*, \dot{x}^*, u, \dot{u}, u^*, \dot{u}^*; \alpha_1) + F(t, x, \dot{x}, x^*, \dot{x}^*, u, \dot{u}, u^*, \dot{u}^*; \alpha_2)
\]

for all \( \alpha_1, \alpha_2 \in \mathbb{R}^n \), and

\[
F(t, x, \dot{x}, x^*, \dot{x}^*, u, \dot{u}, u^*, \dot{u}^*; a\alpha) = aF(t, x, \dot{x}, x^*, \dot{x}^*, u, \dot{u}, u^*, \dot{u}^*; \alpha)
\]

for all \( a \in \mathbb{R} \) with \( a \geq 0 \) and \( \alpha \in \mathbb{R}^n \).

From Definition 2.2, by substituting \( a = 0 \), we have

\[
F(t, x, \dot{x}, x^*, \dot{x}^*, u, \dot{u}, u^*, \dot{u}^*; 0) = 0.
\]

**Definition 2.3.** A system of functions

\[
(\int_a^b f^i(t, x, \dot{x}, u, \dot{u}) dt, \int_a^b g(t, x, \dot{x}, u, \dot{u}) dt)
\]

is said to be \( F \)-invex at \((y, u') \in X_0 \times U_0\) if there exist differentiable functions \( \eta : I \times X_0 \times X_0 \times U_0 \times U_0 \rightarrow \mathbb{R}^n \) with \( \eta(t, x, y, u, u') = 0 \) at \( t \) if \( x = y \) and \( \xi : I \times X_0 \times X_0 \times U_0 \times U_0 \rightarrow \mathbb{R}^n \) with \( \alpha_i : I \times X_0 \times X_0 \times U_0 \times U_0 \rightarrow \mathbb{R}^l \setminus \{0\} \) such that, for each \( x, y \in X_0 \) and \( u, u' \in U_0 \),

\[
\int_a^b f^i(t, x, \dot{x}, u, \dot{u}) dt - \int_a^b f^i(t, y, \dot{y}, u', \dot{u}') dt \geq \int_a^b F(t, x, \dot{x}, y, \dot{y}, u, \dot{u}, u', \dot{u}'; \alpha(t, y, \dot{y}, u, u')) \times \left[ f^i_x(t, y, \dot{y}, u, u') \times \eta(t, y, \dot{y}, u, u') \\
+ \frac{d}{dt} \eta(t, y, \dot{y}, u, u') \times f^i_y(t, y, \dot{y}, u, u') + \xi(t, y, \dot{y}, u, u') \times f^i_u(t, y, \dot{y}, u, u') \right] dt \\
+ p_1 \int_a^b d^2(t, x, y, u, u') dt.
\]

and

\[
- \int_a^b g(t, x, \dot{x}, u, \dot{u}) dt
\]
\[ F(t, x, \dot{x}, y, \dot{y}, u, \dot{u}, u', \dot{u}'); \alpha(t, y, \dot{y}, u, u') \times \left[ g^i_x(t, y, \dot{y}, u, u') \times \eta(t, y, \dot{y}, u, u') \right. \\
\left. + \frac{d}{dt} \eta(t, y, \dot{y}, u, u') \times g^i_x(t, y, \dot{y}, u, u') \right] + \xi(t, y, \dot{y}, u, u') \times g^i_u(t, y, \dot{y}, u, u') \right] \frac{dt}{d} + \rho^2 \int_a^b d^2(t, x, y, u, u') dt. \]

for all \( i \in K \).

If in the above definition, the two inequalities are strict, then \((f, g)\) is said to be \(F\)-strict-invex.

For every \( \zeta \in \mathbb{R}^k_+ \), we consider the following auxiliary problem:

\[(M_P_{\zeta}) \quad \min \left( \int_a^b p^1(t, x, \dot{x}, u, \dot{u}) - \zeta_1 q^1(t, x, \dot{x}, u, \dot{u}) + (x^T B_1 x) \frac{dt}{d}, \cdots, \right. \\
\left. \int_a^b p^k(t, x, \dot{x}, u, \dot{u}) - \zeta_k q^k(t, x, \dot{x}, u, \dot{u}) + (x^T B_k x) \frac{dt}{d} \right) \]

The following lemma can be found in [2].

**Lemma 2.1.** A vector \((x^*, u^*)\) is an efficient solution for \((MFP)\) if and only if there exists \( \zeta^* \in \mathbb{R}^n_+ \) such that \((x^*, u^*)\) is an efficient solution for \((M_P_{\zeta^*})\), and

\[ \zeta^*_i = \frac{\int_a^b \left( p^1(t, x^*, \dot{x}^*, u^*, \dot{u}^*) + (x^*^T B_1 x^*) \frac{dt}{d} \right)}{\int_a^b q^1(t, x^*, \dot{x}^*, u^*, \dot{u}^*) dt}, \quad i = 1, 2, \cdots, k. \]

**Lemma 2.2** ([14] (Generalized Schwartz Inequality)). Let \( B \) a positive semidefinite symmetric matrix of order \( n \). Then, for all \( x, y \in \mathbb{R}^n \),

\[ (x^T B y) \leq \left( x^T B x \right) \frac{1}{2} \left( y^T B y \right) \frac{1}{2}. \]

### 3. Wolfe type vector duality

We present the following Wolfe type multiobjective variational dual problem for \((M_P_{\zeta^*})\):

\[ (WD) \quad \max \left( \int_a^b \left( p^1(t, y, \dot{y}, u', \dot{u}') - \zeta_1 q^1(t, y, \dot{y}, u', \dot{u}') + y^T B_1 v_1 + \mu^T g(t, y, \dot{y}, u', \dot{u}') \right) dt, \right. \\
\left. \cdots, \int_a^b p^k(t, y, \dot{y}, u', \dot{u}') - \zeta_k q^k(t, y, \dot{y}, u', \dot{u}') + y^T B_k v_k + \mu^T g(t, y, \dot{y}, u', \dot{u}') \right) dt \]

subject to \( y(a) = \alpha, \quad y(b) = \beta, \)

\[ \sum_{i=1}^k \lambda_i \{ \eta(\cdot) \times [(\tilde{p}^i_x(y, u', \dot{u}') + B_i v_i - \zeta^*_i \tilde{q}^i_x(y, u', \dot{u}') + \mu^T \tilde{g}_x(y, u', \dot{u}')]) \right. \\
\left. + \frac{d}{dt} \eta(\cdot) \times [\tilde{p}^i_x(y, u') - \zeta^*_i \tilde{q}^i_x(y, u')] + \mu^T \tilde{g}_x(y, u')] \right. \\
\left. + \xi(\cdot) \times [\tilde{p}^i_u(y, u') - \zeta^*_i \tilde{q}^i_u(y, u') + \mu^T \tilde{g}_u(y, u')] \} \right) dt = 0, \tag{3.1} \]

\[ v_i^T B_i v_i \leq 1, \quad i \in K, \quad \mu \geq 0. \tag{3.2} \]

where \( \lambda_i \geq 0 \) for \( i \in K \) with \( \sum_{i=1}^k \lambda_i = 1 \) and \( \mu \geq 0 \).
Theorem 3.1 (Weak Duality). Let \((x, u)\) be a feasible solution for \((MFP)\) and \((y, u', v_1, \ldots, v_k, \mu)\) be a feasible solution for \((WD)\). If

(i) \(\overline{p}^i(x, u) + x^T B_i v_1, -\zeta_i \tilde{q}^i(x, u)\) and \(\mu \tilde{g}(x, u)\) are \(F\)-invex for all \(i \in K\);

(ii) \(\sum_{i=1}^k \lambda_i \rho_i^i + \rho^2 \geq 0\);

then the following inequations can not hold at the same time:

\[
\int_a^b p^i(t, x, \dot{x}, u, \dot{u}) - \zeta_i \tilde{q}^i(t, x, \dot{x}, u, \dot{u}) + (x^T B_i x)^{\frac{1}{2}} dt \\
\leq \int_a^b p^j(t, y, \dot{y}, u', \dot{u}') - \zeta_j \tilde{q}^j(t, y, \dot{y}, u', \dot{u}') + y^T B_j v_j + \mu^T g(t, y, \dot{y}, u', \dot{u}') dt \tag{3.3}
\]

for all \(i \in K\) and

\[
\int_a^b p^j(t, y, \dot{y}, u', \dot{u}') - \zeta_j \tilde{q}^j(t, y, \dot{y}, u', \dot{u}') + y^T B_j v_j + \mu^T g(t, y, \dot{y}, u', \dot{u}') dt \tag{3.4}
\]

for some \(j \in K\).

Proof. Suppose contrary that (3.3) and (3.4) hold at the same time. Since \(\lambda_i \geq 0\) and \(\sum_{i=1}^k \lambda_i = 1\), we have

\[
\int_a^b \sum_{i=1}^k \lambda_i [p^i(x, u) - \zeta_i \tilde{q}^i(x, u)] + (x^T B_i x)^{\frac{1}{2}} dt \\
< \int_a^b \sum_{i=1}^k \lambda_i [p^i(y, u') - \zeta_i \tilde{q}^i(y, u')] + y^T B_i v_i + \mu^T \tilde{g}(y, u') dt,
\]

which together with (3.2) and Lemma 2.2 gives

\[
\int_a^b \sum_{i=1}^k \lambda_i [p^i(x, u) - \zeta_i \tilde{q}^i(x, u)] + x^T B_i v_i dt \\
< \int_a^b \sum_{i=1}^k \lambda_i [p^i(y, u') - \zeta_i \tilde{q}^i(y, u')] + y^T B_i v_i + \mu^T \tilde{g}(y, u') dt. \tag{3.5}
\]

Now by the \(F\)-invex condition and the sublinearity of \(F\), we get

\[
\int_a^b \sum_{i=1}^k \lambda_i [p^i(x, u) - \zeta_i \tilde{q}^i(x, u)] + x^T B_i v_i dt \\
- \int_a^b \sum_{i=1}^k \lambda_i [p^i(y, u') - \zeta_i \tilde{q}^i(y, u')] + y^T B_i v_i + \mu^T \tilde{g}(y, u') dt \\
= \int_a^b \sum_{i=1}^k \lambda_i [(p^i(x, u) + x^T B_i v_i - (p^i(y, u') + y^T B_i v_i)] \\
+ (-\zeta_i \tilde{q}^i(x, u) - (-\zeta_i \tilde{q}^i(y, u'))) - \mu^T \tilde{g}(y, u') dt
\]

\[
\begin{align*}
\geq & \int_{a}^{b} F(t, \ldots ; \alpha(t, \dot{y}, u, u')) \times \left\{ \sum_{i=1}^{k} \lambda_i \{ \eta(\cdot) \times [(\overline{p}_i^j(y, u') + B_i v_i)
\end{align*}
\]

\[
- \zeta^*_i \overline{q}_i^j(y, u') + \mu^T \overline{g}_i(y, u') \]
\end{equation*}

\[
+ \frac{d}{dt} \eta(\cdot) \times [\overline{p}_i^j(y, u') - \zeta^*_i \overline{q}_i^j(y, u') + \mu^T \overline{g}_i(y, u')]
\end{equation*}

\[
+ \xi(\cdot) \times [\overline{p}_i^j(y, u') - \zeta^*_i \overline{q}_i^j(y, u') + \mu^T \overline{g}_i(y, u')] \right\} dt
\end{equation*}

\[
+ \left( \sum_{i=1}^{k} \rho_i^1 + \rho^2 \right) \int_{a}^{b} d^2(t, x, y, u, u') dt.
\end{equation*}

It follows from (3.1) and \( \sum_{i=1}^{k} \rho_i^1 + \rho^2 \geq 0 \) that
\[
\int_{a}^{b} \sum_{i=1}^{k} \lambda_i [\overline{p}_i^j(x, u) - \zeta_i \overline{q}_i^j(x, u) + x^T B_i v_i] dt
\end{equation*}

\[
- \int_{a}^{b} \sum_{i=1}^{k} \lambda_i [\overline{p}_i^j(y, u') - \zeta^*_i \overline{q}_i^j(y, u') + y^T B_i v_i + \mu^T \overline{g}(y, u')] dt \geq 0,
\end{equation*}

which contradicts (3.5). This completes the proof.

**Corollary 3.1.** Let \((y^*, u^*, v_1^*, \ldots, v_k^*, \mu^*)\) be a feasible solution for (WD) such that
\[
\int_{a}^{b} \mu^* T g(t, y^*, \dot{y}^*, u^*, \dot{u}^*) dt = 0 \tag{3.6}
\end{equation*}

and
\[
\int (y^* T B_i y^*)^{\frac{1}{2}} dt \leq \int y^* T B_i v_i^* dt \tag{3.7}
\end{equation*}

for each \(i \in K\) and assume that \((y^*, u^*)\) is feasible for \((MP_{\zeta^*})\). If the conditions (i) and (ii) of Theorem 3.1 hold, then \((y^*, u^*)\) is efficient for \((MP_{\zeta^*})\) and \((y^*, u^*, v_1^*, \ldots, v_k^*, \mu^*)\) is efficient for (WD).

**Proof.** Suppose that \((y^*, u^*)\) is not efficient for \((MP_{\zeta^*})\). Then (3.6) and (3.7) imply that
\[
\int_{a}^{b} p_i^j(t, x, \dot{x}, u, \dot{u}) - \zeta_{i} q_i^j(t, y, \dot{y}, u, \dot{u}) + (x^T B_i x)^{\frac{1}{2}} dt
\end{equation*}

\[
\leq \int_{a}^{b} p_i^j(t, y^*, \dot{y}^*, u^*, \dot{u}^*) - \zeta_{i} q_i^j(t, y^*, \dot{y}^*, u^*, \dot{u}^*)
\end{equation*}

\[
+ y^* T B_i v_i^* + \mu^* T g(t, y^*, \dot{y}^*, u^*, \dot{u}^*) dt
\end{equation*}

for all \(i \in K\) and
\[
\int_{a}^{b} p_i^j(t, x, \dot{x}, u, \dot{u}) - \zeta_{i} q_i^j(t, y, \dot{y}, u, \dot{u}) + (x^T B_j x)^{\frac{1}{2}} dt
\end{equation*}

\[
< \int_{a}^{b} p_i^j(t, y^*, \dot{y}^*, u^*, \dot{u}^*) - \zeta_{i} q_i^j(t, y^*, \dot{y}^*, u^*, \dot{u}^*)
\end{equation*}
for some \( j \in K \). Since \( (y^*, u^*, v_1^*, \ldots, v_k^*, \mu^*) \) is feasible for \((WD)\) and \((x, u)\) is feasible for \((MP_{\mathcal{C}^*})\), it is a contradiction to Theorem 3.1. Thus, \((y^*, u^*)\) is efficient for \((MP_{\mathcal{C}^*})\). Similarly, we can prove that \((y^*, u^*, v_1^*, \ldots, v_k^*, \mu^*)\) is efficient for \((WD)\). This completes the proof.

**Theorem 3.2** (Strong Duality). Let \((x^*, u^*)\) be an efficient solution for \((MP_{\mathcal{C}^*})\). If the following constraint qualification

\[
\int_a^b \sum_{i=1}^k \lambda_i [\tilde{p}^i(x, u) - \zeta_i q^i(x, u) + x^T B_i v_i]\, dt
\]

\[
- \int_a^b \sum_{i=1}^k \lambda_i [\tilde{q}^i(y, u') - \zeta_i^* q^i(y, u') + y^T B_i v_i + \mu^T \tilde{\gamma}(y, u')]\, dt \geq 0
\]

holds, then there exist \( \lambda^* \in \mathbb{R}^k \) and \( \mu^* \in \mathbb{R}^m \) such that

\((x^*, u^*, \lambda^*, v_1^*, \ldots, v_k^*, \mu^*)\)

is a feasible solution of \((WD)\). Furthermore, if all the conditions of Theorem 3.1 are satisfied, then

\((x^*, u^*, \lambda^*, v_1^*, \ldots, v_k^*, \mu^*)\)

is efficient for \((WD)\).

**Proof.** The proof is similar as the one of Kim and Kim [9] and so we omit it here.

**Theorem 3.3** (Converse Duality). Let \((x^*, u^*)\) be an efficient solution for \((MP_{\mathcal{C}^*})\) and

\((\tilde{y}, \tilde{u}, \lambda, v_1, \ldots, v_k, \mu)\)

be an efficient solution for \((WD)\) such that

\[
\int_a^b \sum_{i=1}^k \lambda_i [\tilde{p}^i(t, x^*, \dot{x}^*, u^*, \dot{u}^*) - \zeta_i q^i(t, x^*, \dot{x}^*, u^*, \dot{u}^*) + (x^* T B_i x^*)^{\frac{1}{2}}]\, dt
\]

\[
\leq \int_a^b \sum_{i=1}^k \lambda_i [\tilde{q}^i(t, \tilde{y}, \dot{\tilde{y}}, \tilde{u}, \dot{\tilde{u}}) - \zeta_i^* q^i(t, \tilde{y}, \dot{\tilde{y}}, \tilde{u}, \dot{\tilde{u}}) + \tilde{y}^T B_i v_i + \mu^T \tilde{g}(t, \tilde{y}, \dot{\tilde{y}}, \tilde{u}, \dot{\tilde{u}})]\, dt.
\]

If

\[
\tilde{p}^i(x, u) + x^T B_i v_1, -\zeta_i \tilde{q}^i(x, u), \mu \tilde{g}(x, u), \quad i \in K
\]

are \( F \)-strict-invex with \( \sum_{i=1}^k \lambda_i p^i_l + \rho^2 \geq 0 \), then \((x^*, u^*) = (y, u)\).

**Proof.** Suppose that \((x^*, u^*) \neq (y, u)\). By (3.2) and Lemma 2.2, it follows from (3.8) that

\[
\int_a^b \sum_{i=1}^k \lambda_i [\tilde{p}^i(x^*, u^*) - \zeta_i q^i(x^*, u^*) + x^* T B_i v_i]\, dt
\]
\begin{align*}
\leq \int_{a}^{b} \sum_{i=1}^{k} \lambda_i [\tilde{p}^i(y, u) - \zeta_i q^i(y, u) + y^T B_i v_i + \mu^T \tilde{g}(y, u)] dt. \quad (3.9)
\end{align*}

Due to the $F$-strict-invexity and the sublinearity of $F$, we have
\begin{align*}
\int_{a}^{b} \sum_{i=1}^{k} \lambda_i [\tilde{p}^i(x^*, u^*) - \zeta_i q^i(x^*, u^*) + x^T B_i v_i] dt \\
- \int_{a}^{b} \sum_{i=1}^{k} \lambda_i [\tilde{p}^i(y, u) - \zeta_i q^i(y, u) + y^T B_i v_i + \mu^T \tilde{g}(y, u)] dt \\
= \int_{a}^{b} \sum_{i=1}^{k} \lambda_i \left\{ [\tilde{p}^i(x^*, u^*) + x^T B_i v_i] - [\tilde{p}^i(y, u) + y^T B_i v_i] \\
+ [-\zeta_i q^i(x^*, u^*) - (-\zeta_i q^i(y, u))] - \mu^T \tilde{g}(y, u) \right\} dt \\
\geq \int_{a}^{b} F(t, \cdots ; \alpha(t, y, \hat{y}, u, u')) \times \left\{ \sum_{i=1}^{k} \lambda_i \left\{ [\tilde{p}^i_1(y, u) + B_i v_i] \\
- \zeta_i q^i_2(y, u) + \mu^T \tilde{g}_q(y, u) \right\} \times \eta(\cdot) \\
+ \frac{d}{dt} \eta(\cdot) \times [\tilde{p}^i_1(y, u) - \zeta_i q^i_2(y, u) + \mu^T \tilde{g}_q(y, u)] \\
+ \xi(\cdot) \times [\tilde{p}^i_1(y, u) - \zeta_i q^i_2(y, u) + \mu^T \tilde{g}_q(y, u)] \right\} dt \\
+ \left( \sum_{i=1}^{k} \rho_i^1 + \rho^2 \right) \int_{a}^{b} d^2(t, x, y, u, u') dt
\end{align*}

and so
\begin{align*}
\int_{a}^{b} \sum_{i=1}^{k} \lambda_i [\tilde{p}^i(x^*, u^*) - \zeta_i q^i(x^*, u^*) + x^T B_i v_i] dt \\
- \int_{a}^{b} \sum_{i=1}^{k} \lambda_i [\tilde{p}^i(y, u) - \zeta_i q^i(y, u) + y^T B_i v_i + \mu^T \tilde{g}(y, u)] dt,
\end{align*}

which is a contradiction to (3.8). Thus, $(x^*, u^*) = (y, u)$.

For each $i \in K$, if
\begin{align*}
\tilde{p}^i(x, u) + x^T B_i v_1, \quad -\zeta_i q^i(x, u), \quad \mu^T \tilde{g}(x, u)
\end{align*}
are $F$-invex with $\sum_{i=1}^{k} \lambda_i \rho_i^1 + \rho^2 > 0$, then by using the similar way, we can prove that $(x^*, u^*) = (y, u)$. This completes the proof.

References


