CONVERGENCE THEOREMS OF COMMON FIXED POINTS
FOR A FINITE FAMILY OF UNIFORMLY
QUASI-LIPSCHITZIAN MAPPINGS
IN CONVEX METRIC SPACES

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Abstract. In this paper, we introduce and study some kind of Ishikawa type iterative schemes with errors to approximate a common fixed point of a finite family of uniformly quasi-Lipschitzian mappings in convex metric spaces. Under some suitable conditions, we prove some convergence theorems concerned with the Ishikawa type iterative schemes with errors to approximate a common fixed point of a finite family of uniformly quasi-Lipschitzian mappings in convex metric spaces. The results presented in the paper generalize, improve and unify some recent results in the literature.

1. Introduction

Let \((X, d)\) be a metric space. A mapping \(T : X \to X\) is called asymptotically nonexpansive if there exists \(k_n \in [1, \infty)\), \(\lim_{n \to \infty} k_n = 1\), such that
\[
d(T^n x, T^n y) \leq k_n d(x, y), \quad n = 0, 1, 2, \ldots
\]
for all \(x, y \in X\). Let \(F(T) = \{x \in X : Tx = x\}\). If \(F(T) \neq \emptyset\), then \(T\) is called asymptotically quasi-nonexpansive if there exists \(k_n \in [1, \infty)\), \(\lim_{n \to \infty} k_n = 1\), such that
\[
d(T^n x, p) \leq k_n d(x, p), \quad n = 0, 1, 2, \ldots
\]
for all \(x \in X\) and \(p \in F(T)\). Moreover, it is uniformly quasi-Lipschitzian if there exists a constant \(L > 0\) such that
\[
d(T^n x, p) \leq Ld(x, p), \quad n = 0, 1, 2, \ldots
\]
for all \(x \in X\) and \(p \in F(T)\).

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Remark 1.1 (See [11]). From the above definitions, if \( F(T) \neq \emptyset \), it follows that an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive, and an asymptotically quasi-nonexpansive mapping must be uniformly quasi-Lipschitzian \( (L = \sup_{n \geq 0} \{k_n\} < \infty) \). However, the inverse relation does not hold.

In recent years, asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings have been studied extensively by many authors. Takahashi [9] introduced a notion of a convex metric space which is a more general space, and each linear normed space is a special example of a convex metric space. Later on, Tian [10] give some sufficient and necessary conditions for an Ishikawa iteration process of asymptotically quasi-nonexpansive mappings to converge to fixed points in convex metric spaces. The case of two mappings was introduced by Das and Debats [2]; it has a direct link with the minimization problem. Recently, Liu [6, 7] obtained some sufficient and necessary conditions for the iterative sequence of asymptotically quasi-nonexpansive mappings with an errors member in Banach spaces. Hafiz and Khan [3], Jeong and Kim [5] investigated the approximation of fixed points of two asymptotically quasi-nonexpansive mappings in Banach spaces and uniformly convex Banach spaces.

Very recently, Wang and Liu [11] gave some sufficient and necessary conditions for the Ishikawa type iteration process with errors to approximate the fixed point of two uniformly quasi-Lipschitzian mappings in convex metric spaces, which generalized and unified some corresponding results in [3, 5, 6, 10].

Motivated and inspired by the works mentioned above, in this paper, we present some further findings about the Ishikawa type iterative schemes with errors to approximate a common fixed point of a finite family of uniformly quasi-Lipschitzian mappings in convex metric spaces. Under some suitable conditions, we prove some convergence theorems concerned with the Ishikawa type iterative schemes with errors to approximate a common fixed point of a finite family of uniformly quasi-Lipschitzian mappings in convex metric spaces. The results presented in the paper generalize, improve and unify some recent results in [1, 3, 5, 6, 7, 10, 11].

2. Preliminaries

Now, we recall the following iterative processes due to Mann, Isikawa, and Xu, respectively.

(I) The Mann iteration method [8] is defined as follows: for a convex subset \( K \) of a Banach space \( X \) and a mapping \( T \) from \( K \) into itself, the sequence \( \{x_n\} \) in \( X \) is defined by

\[
x_0 \in K \quad \quad x_{n+1} = (1 - c_n)x_n + c_nTx_n, \quad n \geq 1,
\]

where \( \{c_n\}_{n=0}^{\infty} \) is a real sequence satisfying the following conditions: (i) \( 0 \leq c_n < 1 \) for all \( n \geq 1 \); (ii) \( \sum_{n=0}^{\infty} c_n = \infty \); (iii) \( \lim_{n \to \infty} c_n = 0 \).
(II) Ishikawa iteration process \[4\] is defined as follows: for a convex subset \(K\) of a Banach space \(X\) and a mapping \(T\) from \(K\) into itself, the sequence \(\{x_n\}\) in \(X\) is defined by
\[
x_0 \in K, \\
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n, \\
y_n = (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 0,
\]
where \(\{\alpha_n\}\) and \(\{\beta_n\}\) are two sequences in \([0, 1]\) satisfying the conditions \(0 \leq \alpha_n \leq \beta_n \leq 1\) for all \(n \geq 0\), \(\lim_{n \to \infty} \beta_n = 0\) and \(\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty\).

(III) Ishikawa iteration process with errors \[12\] is defined as follows: for a nonempty convex subset \(K\) of a Banach space \(X\) and a mapping \(T : K \to K\), the sequence \(\{x_n\}\) in \(K\) is defined by
\[
x_0 \in K, \\
x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_nTy_n + \gamma_nu_n, \\
y_n = (1 - \beta_n - \delta_n)x_n + \beta_nTx_n + \delta_nv_n, \quad n \geq 0,
\]
where \(\{\|u_n\|\}\) and \(\{\|v_n\|\}\) are two sequences in \(K\) and \(\{\alpha_n\}\), \(\{\beta_n\}\), \(\{\gamma_n\}\) and \(\{\delta_n\}\) are four real sequences in \([0, 1]\) satisfying certain restrictions.

**Definition 2.1** (See \[10\]). Let \((X, d)\) be a metric space, \(I = [0, 1]\), \(\{a_n\}\), \(\{b_n\}\) and \(\{c_n\}\) be real sequences in \([0, 1]\) with \(a_n + b_n + c_n = 1\) for all \(n = 0, 1, 2, \cdots\). A mapping \(W : X^3 \times I^3 \to X\) is said to be a convex structure on \(X\) if it satisfies the following condition:
\[
d(W(x, y, z, a_n, b_n, c_n), u) \leq a_n d(x, u) + b_n d(y, u) + c_n d(z, u)
\]
for all \((x, y, z, a_n, b_n, c_n) \in X^3 \times I^3\) and \(u \in X\). If \((X, d)\) is a metric with a convex structure \(W\), then \((X, d)\) is called a convex metric space.

**Definition 2.2.** Let \((X, d)\) be a convex metric space and \(E\) be a nonempty subset of \(X\). We say that \(E\) is a convex subset of \(X\) if \(W(x, y, a, b, c) \in E\) for all \((x, y, z, a, b, c) \in E^3 \times I^3\).

**Definition 2.3.** Let \((X, d)\) be a convex metric space with a convex structure \(W : X^3 \times I^3 \to X\) and \(E\) be a nonempty convex subset of \(X\). Let \(T_i : E \to E\) be uniformly quasi-Lipschitzian mappings with \(L_i > 0\) for \(i = 1, 2, \cdots, N\). Let \(\{a_n\}\), \(\{b_n\}\), \(\{c_n\}\), \(\{a'_n\}\), \(\{b'_n\}\) and \(\{c'_n\}\) be six sequences in \([0, 1]\) with \(a_n + b_n + c_n = a'_n + b'_n + c'_n = 1\), \(n = 0, 1, 2, \cdots\), for any given \(x_0 \in E\), define a sequence \(\{x_n\}\) as follows:
\[
\begin{align*}
x_{n+1} &= W(x_n, T_{n+2}^n y_n, u_n, a'_n, b'_n, c'_n), \\
y_n &= W(x_n, T_{n+1}^n x_n, a_n, b_n, c_n),
\end{align*}
\]
where \(\{u_n\}\) and \(\{v_n\}\) are two sequences in \(E\) and \(T_i = T_{i(mod N)}\) for \(i = 1, 2, \cdots\). Then \(\{x_n\}\) is called the Ishikawa type iteration process with errors for a finite family of uniformly quasi-Lipschitzian mappings \(T_i\) for \(i = 1, 2, \cdots, N\).
Remark 2.1. Note that the iteration processes (I), (II) and (III) mentioned above and some iteration processes considered in [1, 3, 5, 6, 10] can be obtained from the above process as special cases by suitably choosing the spaces and the mappings.

In the sequel, we shall need the following lemmas.

Lemma 2.1 (See [6]). Let the nonnegative sequences \( \{p_n\} \), \( \{q_n\} \) and \( \{r_n\} \) satisfy that
\[
p_{n+1} \leq (1 + q_n)p_n + r_n, \quad n \geq 0
\]
and
\[
\sum_{n=1}^{\infty} q_n < \infty, \quad \sum_{n=1}^{\infty} r_n < \infty.
\]
Then (i) \( \lim_{n \to \infty} p_n \) exist; (ii) if \( \lim \inf p_n = 0 \), then \( \lim_{n \to \infty} p_n = 0 \).

Lemma 2.2. Let \((X, d)\) be a convex metric space and \(E\) be a nonempty subset of \(X\). Let \(T_i : E \to E\) be uniformly quasi-Lipschitzian mappings with \(L_i > 0\) for \(i = 1, 2, \cdots, N\) such that \(F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset\). Then there exists a constant \(L \geq 1\) such that
\[
d(T^n_i x, p) \leq L d(x, p), \quad \forall x \in E, \ p \in F, \ n = 0, 1, 2, \cdots.
\]
Proof. For each \(n = 0, 1, 2, \cdots\) and \(i = 1, 2, \cdots, N\), we have
\[
d(T^n_i x, p) \leq L_i d(x, p) \leq L d(x, p), \quad \forall x \in E, \ p \in F,
\]
where \(L = \max_{1 \leq i \leq N} \{L_i\}\). This completes the proof.

Lemma 2.3. Let \(E\) be a nonempty closed convex subset of complete convex metric space \(X\), \(T_i : E \to E\) be uniformly quasi-Lipschitzian mappings with \(L_i > 0\) for \(i = 1, 2, \cdots\) such that \(F = \bigcap_{i=1}^{N} F(T_i) \) is nonempty and bounded. Let \(\{x_n\}\) be a sequence defined by (2.1), in which \(\{u_n\}\) and \(\{v_n\}\) are two bounded sequences. Let \(\{a_n\}\), \(\{b_n\}\), \(\{c_n\}\), \(\{a'_n\}\), \(\{b'_n\}\) and \(\{c'_n\}\) be six sequences in \([0, 1]\) with \(a_n + b_n + c_n = a'_n + b'_n + c'_n = 1\), \(n = 0, 1, 2, \cdots\), satisfying
\[
\sum_{n=0}^{\infty} (b'_n + c'_n) < \infty.
\]
Then
(i) for any \(p \in F\) and \(n = 0, 1, 2, \cdots\),
\[
d(x_{n+1}, p) \leq (1 + b'_n L(1 + L)) d(x_n, p) + M \eta_n, \tag{2.2}
\]
where \(L = \max_{1 \leq i \leq N} \{L_i\}\), \(\eta_n = (b'_n + c'_n)\) for \(n = 0, 1, 2, \cdots\) and
\[
M = \sup_{p \in F, \ n \geq 0} \{d(u_n, p) + L d(v_n, p)\};
\]
(ii) for any \(p \in F\) and \(n = 0, 1, 2, \cdots\),
\[
d(x_{n+m}, p) \leq M_1 d(x_n, p) + M M_1 \sum_{k=n}^{n+m-1} \eta_k, \tag{2.3}
\]
where \(M_1 = e^{L(1+L) \sum_{k=0}^{\infty} b'_k}\).
Proof. (i) For any $p \in F$, it follows from (2.1) and Lemma 2.1 that
\[ d(x_{n+1}, p) = d(W(x_n, T_{n+2}^{m}y_n, n, a_n', b_n', c_n'), p) \]
\[ \leq a_n' d(x_n, p) + b_n' d(T_{n+2}^{m}y_n, p) + c_n' d(u_n, p) \]
\[ \leq a_n' d(x_n, p) + b_n' L d(y_n, p) + c_n' d(u_n, p) \quad (2.4) \]
and
\[ d(y_n, p) = d(W(x_n, T_{n+1}^{m}x_n, v_n, a_n, b_n, c_n), p) \]
\[ \leq a_n d(x_n, p) + b_n d(T_{n+1}^{m}x_n, p) + c_n d(v_n, p) \]
\[ \leq a_n d(x_n, p) + b_n L d(x_n, p) + c_n d(v_n, p) \]
\[ = (a_n + b_n L) d(x_n, p) + c_n d(v_n, p). \quad (2.5) \]
By (2.4) and (2.5), we have
\[ d(x_{n+1}, p) \leq a_n' d(x_n, p) + b_n' L [(a_n + b_n L) d(x_n, p) + c_n d(v_n, p)] + c_n' d(u_n, p) \]
\[ \leq [1 + b_n' L (1 + L)] d(x_n, p) + c_n' d(u_n, p) + b_n' c_n L d(v_n, p) \]
\[ \leq (1 + b_n' L (1 + L)) d(x_n, p) + (d(u_n, p) + L d(v_n, p)) (b_n' + c_n') \]
\[ \leq (1 + b_n' L (1 + L)) d(x_n, p) + M \eta_n \]
and so (2.2) holds.

(ii) It is well known that $1 + x \leq e^x$ for all $x \geq 0$. Using this fact, for any $p \in F$ and $m, n \geq 0$, it follows from (2.2) that
\[ d(x_{n+m}, p) \leq (1 + b_{n+m} L (1 + L)) d(x_{n+m-1}, p) + M \eta_{n+m-1} \]
\[ \leq e^{b_{n+m-1} L (1 + L)} d(x_{n+m-1}, p) + M \eta_{n+m-1} \]
\[ \leq e^{b_{n+m-1} L (1 + L)} [(1 + b_{n+m-2} L (1 + L)) d(x_{n+m-2}, p) + M \eta_{n+m-2}] \]
\[ + M \eta_{n+m-1} \]
\[ \leq e^{(b_{n+m-1} + b_{n+m-2}) L (1 + L)} d(x_{n+m-2}, p) + M [\eta_{n+m-2} + \eta_{n+m-1}] \]
\[ \quad \ldots \]
\[ \leq M_1 d(x_n, p) + M_1 M \sum_{k=n}^{n+m-1} \eta_k \]
and so (2.3) holds. This completes the proof.

3. Main Results

Theorem 3.1. Let $E$ be a nonempty closed convex subset of complete convex metric space $X$, $T_i : E \rightarrow E$ be uniformly quasi-Lipschitzian mappings with $L_i > 0$ for $i = 1, 2, \ldots, N$ such that $F = \bigcap_{i=1}^{N} F(T_i)$ is nonempty and bounded. Let \{x_n\} be a sequence defined by (2.1), in which \{u_n\} and \{v_n\} are two bounded sequences. Let \{a_n\}, \{b_n\}, \{c_n\}, \{a_n'\}, \{b_n'\} and \{c_n'\} be six sequences in [0, 1] with $a_n + b_n + c_n = a_n' + b_n' + c_n' = 1$, $n = 0, 1, 2, \ldots$, satisfying
\[ \sum_{n=0}^{\infty} (b_n' + c_n') \leq \infty. \]
Then \( \{x_n\} \) converges to a common fixed point of \( p \in F \) if and only if
\[
\liminf_{n \to \infty} d(x_n, F) = 0,
\]
where \( d(x, F) = \inf\{d(x, p) : p \in F\} \).

**Proof.** The necessity of the conditions is obvious. Thus, we only need to prove the sufficiency. By Lemma 2.3 (i), we have
\[
 d(x_{n+1}, F) \leq (1 + b'_n L(1 + L)) d(x_n, F) + M \eta_n, \quad n = 0, 1, 2, \ldots .
\]  
(3.1)
Since
\[
\sum_{n=0}^{\infty} \eta_n = \sum_{n=0}^{\infty} (b'_n + c'_n) < \infty,
\]
it follows from (3.1) and Lemma 2.1 that \( \lim_{n \to \infty} d(x_n, F) \) exists. Now \( \liminf_{n \to \infty} d(x_n, F) = 0 \) implies that \( \lim_{n \to \infty} d(x_n, F) = 0 \).

Next we show that \( \{x_n\} \) is a Cauchy sequence. For any \( \varepsilon > 0 \), there exists a positive integer \( N_0 \) such that, for all \( n \geq N_0 \),
\[
 d(x_n, F) \leq \frac{\varepsilon}{4M_1}, \sum_{n=N_0}^{\infty} \eta_n \leq \frac{\varepsilon}{4M_0 M_1}.
\]
(3.2)
In particular, there exists a \( p_1 \in F \) and a positive integer \( N_1 > N_0 \) such that
\[
 d(x_{N_1}, p_1) \leq \frac{\varepsilon}{4M_1}.
\]
For any positive integers \( n, m \) with \( n > N_1 \), By (3.2) and Lemma 2.3 (ii), we have
\[
 d(x_{n+m}, x_n) \leq d(x_{n+m}, p_1) + d(p_1, x_n)
\]
\[
 \leq M_1 d(x_{N_1}, p_1) + M_1 M \sum_{k=N_1}^{n+m-1} \eta_k + M_1 d(x_{N_1}, p_1) + M_1 M \sum_{k=N_1}^{n-1} \eta_k
\]
\[
 \leq 2M_1 \frac{\varepsilon}{4M_1} + 2M_1 M \frac{\varepsilon}{4M_1 M}
\]
\[
 = \varepsilon.
\]
This shows that \( \{x_n\} \) is a Cauchy sequence in \( E \). Let \( \lim_{n \to \infty} x_n = p^* \in E \).

Finally, we show that \( p^* \in F \). To this end, we only need to prove that \( F \) is closed because
\[
 d(p^*, F) = \lim_{n \to \infty} d(x_n, F) = 0.
\]
Let \( p_n \in F \) be a sequence such that \( \lim_{n \to \infty} p_n = p' \). We show that \( p' \in F \). In fact, for any \( i = 1, 2, \ldots , N \),
\[
 d(p', T_ip') \leq d(p', p_n) + d(p_n, T_ip')
\]
\[
 = d(p', p_n) + d(T_ip_n, T_ip')
\]
\[
 \leq d(p', p_n) + L d(p_n, p')
\]
and this implies that
\[
 d(p', T_ip') = 0, \quad i = 1, 2, \ldots , N.
\]
Thus, \( p' \in F \) and so \( F \) is closed. This completes the proof.
By Theorem 3.1, we can get the following results.

**Theorem 3.2.** Let $E$ be a nonempty closed convex subset of complete convex metric space $X$, $T_i : E \to E$ be asymptotically quasi-nonexpansive mappings with $k^i_n > 0$ for $i = 1, 2, \ldots, N$ such that $F = \bigcap_{i=1}^{N} F(T_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence defined by (2.1), in which $\{u_n\}$ and $\{v_n\}$ are two bounded sequences. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ be six sequences in $[0, 1]$ with $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, $n = 0, 1, 2, \cdots$, satisfying
\[
\sum_{n=0}^{\infty} (b'_n + c'_n) < \infty.
\]
Then $\{x_n\}$ converges to a common fixed point of $p \in F$ if and only if
\[
\liminf_{n \to \infty} d(x_n, F) = 0.
\]

**Theorem 3.3.** Let $E$ be a nonempty closed convex subset of complete convex metric space $X$, $T_i : E \to E$ be asymptotically nonexpansive mappings with $k^i_n > 0$ for $i = 1, 2, \cdots, N$ such that $F = \bigcap_{i=1}^{N} F(T_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence defined by (2.1), in which $\{u_n\}$ and $\{v_n\}$ are two bounded sequences. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ be six sequences in $[0, 1]$ with $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, $n = 0, 1, 2, \cdots$, satisfying
\[
\sum_{n=0}^{\infty} (b'_n + c'_n) < \infty.
\]
Then $\{x_n\}$ converges to a common fixed point of $p \in F$ if and only if
\[
\liminf_{n \to \infty} d(x_n, F) = 0.
\]

**Remark 3.1.** Theorems 3.1-3.3 generalize, improve and unify some recent results in [1, 3, 5, 6, 7, 10, 11].

**References**

