FIFTH-ORDER ITERATIVE SCHEME USING
DECOMPOSITION TECHNIQUE

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ABSTRACT. In this paper, we suggest and analyze a new three-step method for solving nonlinear equations using the decomposition technique which is mainly due to Noor et al. [6]. We show that this new iterative method has fifth-order of convergence. To demonstrate the efficiency and performance of the new method, we tested several numerical examples and results are shown in the Table-1.

1. Introduction

In recent years, much attention has been given to develop several iterative methods for solving nonlinear equations, see [2-6] and the references therein. These methods can be classified as one-step, two-step and three-step methods. In [2] Chun has proposed and studied several one-step and two-step iterative methods with higher-order of convergence by using the decomposition technique of Adomian [1]. In the methods of Chun [2], higher-order differential derivatives are involved, which is a serious drawback. To overcome this draw back, we suggest and analyze a family of multi-step methods for solving nonlinear equations using a different type of decomposition, which does not involve the high-order differentials of the function. This new decomposition is mainly due to Noor et al. [6].

2. Iterative Methods

Consider the nonlinear equation

\[ f(x) = 0. \] (2.1)
We assume that $\alpha$ is a simple root of (2.1) and $\gamma$ is an initial guess sufficiently close to $\alpha$. We can rewrite the nonlinear equation (2.1) as a coupled system:

$$f(\gamma) + (x - \gamma) \frac{2 f'(\gamma) f'(x)}{f'(\gamma) + f'(x)} + g(x) = 0,$$

(2.2)

$$g(x) = f(x) - f(\gamma) - (x - \gamma) \frac{2 f'(\gamma) f'(x)}{f'(\gamma) + f'(x)},$$

(2.3)

where $\gamma$ is the initial approximation for a zero of (2.1).

We can rewrite (2.2) in the following form:

$$x = \gamma - \frac{f(\gamma) (f'(\gamma) + f'(x))}{2 f'(\gamma) f'(x)} - \frac{g(x) (f'(\gamma) + f'(x))}{2 f'(\gamma) f'(x)}.$$  

(2.4)

We can also rewrite (2.4) as:

$$x = c + N(x),$$

(2.5)

where

$$c = \gamma - \frac{f(\gamma) (f'(\gamma) + f'(x))}{2 f'(\gamma) f'(x)},$$

(2.6)

and

$$N(x) = - \frac{g(x) (f'(\gamma) + f'(x))}{2 f'(\gamma) f'(x)}.$$  

(2.7)

Here $N(x)$ is a linear operator.

We note that if $x_0$ is the initial guess, then from (2.2) and (2.3), we have

$$f(x_0) = g(x_0).$$

(2.8)

As in [6], the solution of (2.5) has the series form,

$$x = \sum_{i=0}^{\infty} x_i.$$  

(2.9)

The nonlinear operator $N(x)$ can be decomposed as

$$N \left( \sum_{i=0}^{\infty} x_i \right) = N(x_0) + \sum_{i=1}^{\infty} \left\{ N \left( \sum_{j=0}^{i} x_j \right) \right\}.$$  

(2.10)

Combining (2.5), (2.9) and (2.10), we have

$$\sum_{i=0}^{\infty} x_i = c + N(x_0) + \sum_{i=1}^{\infty} \left\{ N \left( \sum_{j=0}^{i} x_j \right) \right\}.$$  

(2.11)

Thus we have the following iterative scheme:

$$x_0 = c,$$

$$x_1 = N(x_0),$$

$$x_2 = N(x_0 + x_1),$$

$$\vdots$$

$$x_{n+1} = N(x_0 + x_1 + \ldots + x_n), \quad n = 1, 2, \ldots$$

(2.12)
Then
\[ x_1 + x_2 + \ldots + x_{n+1} = N(x_0) + N(x_0 + x_1) + \ldots + N(x_0 + x_1 + \ldots + x_n), \quad n = 0, 1, 2, \ldots, \]
and
\[ x = c + \sum_{i=1}^{\infty} x_i. \quad (2.13) \]

From (2.6), (2.7), (2.8) and (2.11), we have
\[ x_0 \approx c = \gamma - \frac{f(x_0) (f'(\gamma) + f'(x_0))}{2f'(\gamma)f'(x)}, \quad (2.14) \]
and
\[ x_1 = N(x_0) = -\frac{g(x_0) (f'(\gamma) + f'(x_0))}{2f'(\gamma)f'(x_0)} = -\frac{f(x_0) (f'(\gamma) + f'(x_0))}{2f'(\gamma)f'(x_0)}. \quad (2.15) \]

It follows from (2.12), (2.13) and (2.14) that
\[ x \approx x_0 = c = \gamma - \frac{f(x_0) (f'(\gamma) + f'(x))}{2f'(\gamma)f'(x)}. \]

This enables us to suggest the following method for solving the nonlinear equation (2.1).

**Algorithm 1.** For the given \( x_0 \) compute the approximate solution \( x_{n+1} \) by the iterative schemes:
\[ z_n = x_n = \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \]
\[ x_{n+1} = x_n = \frac{f(x_n) (f'(x_n) + f'(z_n))}{2f'(x_n)f'(z_n)}, \quad (2.16) \]
which is known as Özban method [7] and is cubically convergent.

Again by using (2.12), (2.13), (2.14) and (2.15), we conclude that
\[ x \approx c + x_1 \]
\[ = x_0 + N(x_0) \]
\[ = \gamma - \frac{f(x_0) (f'(\gamma) + f'(x))}{2f'(\gamma)f'(x)} - \frac{f(x_0) (f'(\gamma) + f'(x_0))}{2f'(\gamma)f'(x_0)}. \quad (2.17) \]

Using (2.17), we can suggest the following three-step iterative method for solving nonlinear equation (2.1) as:

**Algorithm 2.** For the given \( x_0 \) compute the approximate solution \( x_{n+1} \) by the iterative schemes:
\[ z_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \]
\[ y_n = x_n - \frac{f(x_n) (f'(x_n) + f'(z_n))}{2f'(x_n)f'(z_n)}, \]
\[ x_{n+1} = y_n - \frac{f(y_n) (f'(x_n) + f'(y_n))}{2f'(x_n)f'(y_n)}. \]
Also, using (2.7), (2.8) and (2.12), we can calculate
\[
x_2 = N(x_0 + x_1) = -\frac{f(x_0 + x_1) (f'(\gamma) + f'(x_0 + x_1))}{2f'(\gamma)f''(x_0 + x_1)}.
\]
(2.18)

From (2.12), (2.13), (2.14), (2.15) and (2.18), we get
\[
x \approx c + x_1 + x_2
= x_0 + N(x_0) + N(x_0 + x_1)
= \gamma - \frac{f(\gamma) (f'(\gamma) + f'(x))}{2f'(\gamma)f'(x)} - \frac{f(x_0) (f'(\gamma) + f'(x_0))}{2f'(\gamma)f'(x_0)}
- \frac{f(x_0 + x_1) (f'(\gamma) + f'(x_0 + x_1))}{2f'(\gamma)f'(x_0 + x_1)}.
\]
(2.19)

Using this, we can suggest and analyze the following four-step iterative method for solving nonlinear equation (2.1).

**Algorithm 3. (FAH)** For a given \( x_0 \), compute the approximate solution by the iterative schemes:

\[
\begin{align*}
z_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \\
y_n &= x_n - \frac{f(x_n) (f'(x_n) + f'(z_n))}{2f'(x_n)f'(z_n)}, \\
s_n &= -\frac{f(y_n) (f'(x_n) + f'(y_n))}{2f'(x_n)f'(y_n)}, \\
x_{n+1} &= y_n + s_n - \frac{f(y_n + s_n) (f'(x_n) + f'(y_n + s_n))}{2f'(x_n)f'(y_n + s_n)}.
\end{align*}
\]
(3.1)

3. Analysis of Convergence

**Theorem 1.** Assume that the function \( f : D \subseteq \mathbb{R} \to \mathbb{R} \) for an open interval \( D \) has a simple zero \( \alpha \in D \). Let \( f(x) \) be sufficiently smooth in the neighborhood of \( \alpha \), then the order of convergence of the method defined by Algorithm 3 is five.

**Proof.** The iterative scheme is given by
\[
\begin{align*}
z_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \\
y_n &= x_n - \frac{f(x_n) (f'(x_n) + f'(z_n))}{2f'(x_n)f'(z_n)}, \\
s_n &= -\frac{f(y_n) (f'(x_n) + f'(y_n))}{2f'(x_n)f'(y_n)}, \\
x_{n+1} &= y_n + s_n - \frac{f(y_n + s_n) (f'(x_n) + f'(y_n + s_n))}{2f'(x_n)f'(y_n + s_n)}.
\end{align*}
\]
(3.4)

Let \( \alpha \) be a simple zero of \( f \). By Taylor’s expansion, we have,
\[
\begin{align*}
f(x_n) &= f'(\alpha) [e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O(e_n^7)], \\
f'(x_n) &= f'(\alpha) [1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + O(e_n^6)].
\end{align*}
\]
(3.5)

(3.6)
where
\[ c_k = \frac{1}{k!} \frac{f^{(k)}(\alpha)}{f'(\alpha)}, \quad k = 2, 3, \ldots, \text{and} \ e_n = x_n - \alpha. \]

From (3.5) and (3.6), we have
\[
\frac{f(x_n)}{f'(x_n)} = e_n - 2e_n^2 + (2e_n^2 - 2c_e) e_n^3 + (7c_2c_3 - 4e_n^4 - 3c_4) e_n^4 \\
+ (-4c_5 + 10c_2c_4 - 20c_3c_2^2 + 8c_2^3 + 6c_2^4) e_n^5 + O(e_n^6). \tag{3.7}
\]

Using (3.1) and (3.7), we get
\[
z_n = \alpha + c_2e_n^2 + (2c_3 - 2e_n^2)e_n^3 + (3c_4 - 7c_2c_3 + 4e_n^4 + O(e_n^5). \tag{3.8}
\]

Again by Taylor’s series,
\[
f(z_n) = f'(\alpha)e_n^2 + (2c_3 - 2c_2)e_n^3 + (3c_4 - 7c_2c_3 + 4e_n^4 + (4c_5 - 10c_2c_4 \\
- 6c_3^2 + 24c_3c_2^2 - 12c_2^3) e_n^4 + O(e_n^5). \tag{3.9}
\]

and
\[
f'(z_n) = f'(\alpha)[1 + 2c_2e_n^2 + (4c_2c_3 - 4c_2) e_n^3 + (6c_2c_4 - 11c_3c_2^2 + 8c_2^3)e_n^4 \\
+ (8c_2c_5 - 20c_2c_4 + 28c_3c_2^3 - 16c_2^5)e_n^5 + O(e_n^6)] \tag{3.10}
\]

From (3.2), (3.5), (3.6) and (3.10), we get
\[
y_n = \alpha + \frac{1}{2}c_3 e_n^3 + \left( c_4 + c_3^2 - \frac{3}{2}c_2c_3 \right) e_n^4 + \left( \frac{3}{2}e_5 - 2c_2c_4 + \frac{15}{2}c_3c_2^2 \\
- 4c_2^3 - 3c_3^2 \right) e_n^5 + O(e_n^6), \tag{3.11}
\]

implies by Taylor’s series,
\[
f(y_n) = f'(\alpha) \left[ \frac{1}{2}c_3 e_n^3 + \left( c_4 + c_3^2 - \frac{3}{2}c_2c_3 \right) e_n^4 + \left( \frac{3}{2}e_5 - 2c_2c_4 + \frac{15}{2}c_3c_2^2 \\
- 4c_2^3 - 3c_3^2 \right) e_n^5 + O(e_n^6) \right], \tag{3.12}
\]

and
\[
f'(y_n) = f'(\alpha) \left[ 1 + c_2c_3 e_n^2 + (2c_2c_4 + 2c_2^3 - 3c_3c_2^2) e_n^4 + (3c_5 + 4c_2c_4 - 4c_2^3) e_n^4 \\
+ 15c_2c_4^2 - 8c_2^3 - 6c_3c_2^3 \right] e_n^6 + O(e_n^6). \tag{3.13}
\]

Using (3.3), (3.6), (3.12) and (3.13), we obtain
\[
s_n = - \frac{1}{2}c_3 e_n^3 + (-c_4 + c_3^2 + 2c_2c_3) e_n^4 + \left( - \frac{3}{2}e_5 + 3c_2c_4 - 10c_3c_2^2 + 5c_2^4 \\
+ \frac{15}{4}c_3^2 \right) e_n^5 + O(e_n^6). \tag{3.14}
\]

Using (3.11) and (3.14), we have
\[
y_n + s_n = \alpha + \frac{1}{2}c_2c_3 e_n^4 + \left( c_2c_4 - \frac{5}{2}c_3c_2^2 + c_2^4 + \frac{3}{4}c_3^2 \right) e_n^5 + O(e_n^6), \tag{3.15}
\]

which implies by Taylor’s series,
\[
f(y_n + s_n) = f'(\alpha) \left[ \frac{1}{2}c_2c_3 e_n^4 + \left( c_2c_4 - \frac{5}{2}c_3c_2^2 + c_2^4 + \frac{3}{4}c_3^2 \right) e_n^5 + O(e_n^6) \right]. \tag{3.16}
\]
and
\[ f'(y_n + s_n) = f'(\alpha) \left[ 1 + c_3c_2^2c_n^4 + \left( 2c_2^2c_4 - 5c_3c_2^3 + 2c_2^3 + \frac{3}{2}c_2c_3^2 \right)c_n^5 + O(e_n^6) \right]. \] (3.17)

Finally using (3.4), (3.6), (3.16) and (3.17), we obtain
\[ x_{n+1} = \alpha + \frac{1}{2}c_3c_2^2c_n^5 + O(e_n^6), \]
implies
\[ e_{n+1} = \frac{1}{2}c_3c_2^2c_n^5 + O(e_n^6). \]

Hence proved.

4. Numerical Examples

In this section we consider some numerical examples to demonstrate the performance of the new developed iterative method. We compare Chun method [2] (CHU), Noor and Noor method [4] (NR3), Noor and Noor method [3] (NR1) and Noor and Noor method [5] (NR2) with the new developed method (FAH). All the computations for above mentioned methods, are performed using software Maple 9, precision 60 digits and \( \varepsilon = 10^{-18} \) as tolerance and also the following criteria is used for estimating the zero:

(i) \( \delta = |x_{n+1} - x_n| < \varepsilon. \)
(ii) \( |f(x_n)| < \varepsilon. \)
(iii) Maximum numbers of iterations = 500.

We use the following examples for numerical testing and results are given in the Table-1.

\[ \begin{array}{cccc}
\text{Example} & f(x) & \text{Exact Zero} & \\
\hline
f_1 & \sin^2(x) - x^2 + 1 & \alpha = 1.4044916482153411 & \\
f_2 & x^2 - e^x - 3x + 2 & \alpha = 0.2575302854398608 & \\
f_3 & \cos(x) - x & \alpha = 0.7390851332156067 & \\
f_4 & (x - 1)^3 - 1 & \alpha = 2 & \\
f_5 & x^3 - 10 & \alpha = 2.15443469003188372 & \\
f_6 & xe^x - \sin^2(x) + 3\cos(x) + 5 & \alpha = -1.2076478271309191 & \\
f_7 & e^{x^2} + 7x - 30 - 1 & \alpha = 3 & \\
\end{array} \]

<table>
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<th>Table-1</th>
<th>n</th>
<th>( f(x_n) )</th>
<th>( \delta )</th>
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<td>$\delta$</td>
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5. Conclusion

In the Table-1, we observe that our iterative method (FAH) is comparable with all the cited methods and gives better results. The technique and idea of this paper can be developed to higher-order multi-step iterative methods for solving nonlinear equations, as well as a system of nonlinear equations.

References