SOME APPLICATIONS OF QUADRATURE FORMULAS FOR SOLVING NONLINEAR EQUATIONS

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Abstract. In this paper, we suggest and analyze two new two-step iterative methods for solving nonlinear equations using quadrature formulas. We prove that these new methods have cubic convergence. Several examples are given to illustrate the efficiency of these new methods and its comparison with other similar methods. Our results can be considered as an alternative to Newton method and other similar methods.

1. Introduction

In recent years much attention has been given to develop several iterative type methods for solving nonlinear equations. These methods have been suggested by using Taylor’s series, decomposition, homotopy, variational iteration and quadrature formulas, see [1-12]. It is well known that the quadrature rules play an important and significant role in the evaluation of the integrals. It has been shown [3, 5, 9, 10, 12] that these quadrature formulas can be used to develop some iterative methods for solving nonlinear equations. Motivated and inspired by the ongoing activities in this direction, we suggest and analyze two new iterative methods for solving nonlinear equations by using the quadrature formula, see [11]. These methods are implicit-type methods. To implement these methods, we use Newton method as a predictor and then use new methods as a corrector. The resultant methods can be considered predictor-corrector methods or two-step iterative methods. It has been shown that these two-step iterative methods are of cubic convergence under certain conditions. A comparison between these new methods with that of Newton method and other similar methods is given. Several examples are given to illustrate the efficiency and advantage of these two-step methods. Our methods can be considered as an alternative iterative method to recently suggested methods.

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2. Iterative methods

Let us take \( r \) be the simple zero of a sufficiently differentiable function, and consider the numerical solution of the equation \( f(x) = 0 \). Then

\[
f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt.
\]  

(1)

Using the quadrature formula, we have

\[
\int_{x_n}^{x} f'(t)dt = \frac{x-x_n}{4} \left[ f'(x_n) + 3f' \left( \frac{x_n + 2x}{3} \right) \right].
\]  

(2)

From (1) and (2), we have

\[
x = x_n - \frac{4f(x_n)}{f'(x_n) + 3f' \left( \frac{x_n + 2x}{3} \right)}.
\]

This fixed point formulation enables us to suggest the following implicit iterative method.

**Algorithm 2.1.** For a given \( x_0 \), compute the approximate solution \( x_{n+1} \) by the iterative schemes:

\[
x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n) + 3f' \left( \frac{x_n + 2x}{3} \right)}.
\]

We also need the following one-step method which is known as the Newton method for solving nonlinear equations.

**Algorithm 2.2.** For a given \( x_0 \), compute the approximate solution \( x_{n+1} \) by the iterative schemes:

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots
\]

It is well known that Newton method has quadratic convergence.

In order to implement Algorithm 2.1, we use the predictor-corrector technique. Using the Newton method as a predictor, we suggest the following new iterative method, which is the main motivation of this paper.

**Algorithm 2.3.** For a given \( x_0 \), compute the approximate solution \( x_{n+1} \) by the iterative schemes:

**Predictor step.**

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)}.
\]  

(3)

**Corrector step.**

\[
x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n) + 3f' \left( \frac{x_n + 2y_n}{3} \right)}.
\]  

(4)

which is called the two-step method or predictor-corrector method.
In a similar way, using the quadrature formula
\[
\int_{x_n}^{x} f'(t)dt = \frac{x - x_n}{4} \left[ f'(x_n) + 3f' \left( \frac{2x_n + x}{3} \right) \right].
\]
and (1), we have
\[
x = x_n - \frac{4f(x_n)}{f'(x_n) + 3f' \left( \frac{2x_n + x}{3} \right)}.
\]
This fixed point formulation allows us to suggest the following iterative method for solving the nonlinear equation \( f(x) = 0 \) as:

**Algorithm 2.4.** For a given \( x_0 \), compute the approximate solution \( x_{n+1} \) by the iterative schemes:

**Predictor step.**
\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

**Corrector step.**
\[
x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n) + 3f' \left( \frac{2x_n + x}{3} \right)}.
\]
which is called the two-step method or predictor-corrector method.

We now consider the convergence criteria of Algorithm 2.3 using essentially the technique of Noor [8,9] and Corderro and Rorregrosa [3].

**Theorem 2.1.** Let \( r \in I \) be a simple zero of sufficiently differentiable function \( f: \subset R \rightarrow R \) for an open interval \( I \). If \( x_0 \) is sufficiently close to \( r \), then the two-step iterative method defined by Algorithm 2.3 has third-order convergence.

**Proof.** Let \( r \) be a simple zero of \( f \). Since \( f \) is sufficiently differentiable, by expanding \( f(x_n) \) and \( f'(x_n) \) about \( r \), we have
\[
f(x_n) = f'(r)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \ldots] \quad (5)
\]
\[
f'(x_n) = f'(r)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + \ldots], \quad (6)
\]
where \( c_k = \frac{1}{k!} \frac{f^{(k)}(r)}{f'(r)} \), \( k = 1, 2, 3, \ldots \) and \( e_n = x_n - r \).

Now from (5) and (6), we have
\[
\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3) e_n^3 + (7c_2 c_3 - 4c_2^3 - 3c_4) e_n^4 + \ldots \quad (7)
\]
and
\[
y_n = r + c_2 e_n^2 = 2(c_2^2 - c_3) e_n^3 + (7c_2 c_3 - 4c_2^3 - 3c_4) e_n^4 + \ldots \quad (8)
\]
From (8), we obtain
\[
f'(y_n) = f'(r) \left[ 1 + 2c_2(y_n - r) + 3c_3(y_n - r)^2 + 4c_4(y_n - r)^3 + \ldots \right]
= f'(r) \left[ 1 + 2c_2 c_n^2 + 4(c_2^2 - c_3) e_n^3 + \ldots \right]. \quad (9)
\]
From (6) and \( x_n = e_n + r \), we have
\[
\frac{x_n + 2y_n}{3} = r + \frac{1}{3}e_n + \frac{2}{3}c_2 e_n^3 + \left( \frac{4}{3} c_3 - \frac{4}{3} c_2^2 \right) e_n^3 \\
+ \left( -\frac{14}{3}c_2 e_3 + \frac{8}{3} c_2^3 + 2c_4 \right) e_n^4 + \ldots
\] (10)

From (10), we have
\[
f'(x_n) = f'(r) \left\{ 1 + \frac{2}{3}c_2 e_n + \left( \frac{1}{3} c_3 + \frac{4}{3} \right) c_2^2 e_n^2 + \left( 4c_2 c_3 - \frac{8}{3} c_2^3 + \frac{4}{7} \right) e_n^3 \\
+ \left( \frac{16}{3} c_2^4 + \frac{5}{81} c_5 - \frac{32}{3} c_2 c_3 + \frac{44}{9} c_2 c_4 \right) e_n^4 + \ldots \right\}. \tag{11}
\]

From (6), (9) and (11), we have
\[
f'(x_n) + 3f' \left( \frac{x_n + 2y_n}{3} \right) = f'(r) \left\{ 4 + 4c_2 e_n + 4(e_2^2 + c_3) c_2 e_n^2 \\
+ \left( \frac{40}{9} c_4 + 12c_2 c_3 - 8c_2^3 \right) e_n^3 \\
+ \left( 8c_2^2 - 32c_3 c_2^2 + 16c_2^4 + \frac{44}{3} c_2 c_4 + \frac{140}{27} c_5 \right) e_n^4 + \ldots \right\}. \tag{12}
\]

From (4), (12) and \( e_n = x_{n+1} - r \), we have
\[
e_{n+1} = c_2^2 e_n^3 + 3 \left( c_2 c_3 - c_3^2 + \frac{1}{9} c_4 \right) e_n^4 + \ldots,
\]
which shows that Algorithm 2.3 has cubic convergence.

## 3. Numerical Results

We now present some examples to illustrate the efficiency and the comparison of the newly developed two-step iterative methods (Algorithm 2.3 and Algorithm 2.4), see Table 3.1. We compare the Newton Method (NM), the method of Weerakoon and Fernando (WN[12]), The method of Corderro and Torregrosa (CT[3]), method of Ozban (HN[10]), Algorithm 2.3(NR1) and Algorithm 2.4(NR2) introduced in this paper. We used \( \epsilon = 10^{-15} \). The following stopping criteria is used for computer programs:

(i). \( |x_{n+1} - x_n| < \epsilon \),

(ii). \( |f(x_n)| < \epsilon \).

The examples are the same as in Corderos and Torregrosa [3].

\[
\begin{align*}
f_1(x) &= x^3 - 9x^2 + 28x - 30 \\
f_2(x) &= \sin(x) + x \cos(x) \\
f_3(x) &= e^{x^2} - e^{\sqrt{2}x} \\
f_4(x) &= \left( \sin(x) - \frac{x}{2} \right)^2 \\
f_5(x) &= \cos(x) - x \\
f_6(x) &= \tan^{-1}(x) \\
f_7(x) &= (x - 1)^6 - 1
\end{align*}
\]
Some applications of quadrature formulas for solving nonlinear equations

\[ f_8(x) = 4\sin(x) - x + 1 \]

<table>
<thead>
<tr>
<th>Table 1. (Numerical Examples and Comparison)</th>
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<tbody>
<tr>
<td>function</td>
</tr>
<tr>
<td>( f_1, x_0 = 1 )</td>
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<tr>
<td>( f_1, x_0 = 0.5 )</td>
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<tr>
<td>( f_2, x_0 = 1.7 )</td>
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<tr>
<td>( f_3, x_0 = 3 )</td>
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<tr>
<td>( f_4, x_0 = 0.5 )</td>
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<tr>
<td>( f_5, x_0 = 1.5 )</td>
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<tr>
<td>( f_6, x_0 = 0.5 )</td>
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<tr>
<td>( f_7, x_0 = 0.6 )</td>
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<tr>
<td>( f_8, x_0 = 0 )</td>
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<tr>
<td>( f_9, x_0 = 0.5 )</td>
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<tr>
<td>( f_{10}, x_0 = 1 )</td>
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<tr>
<td>( f_7, x_0 = 0.6 )</td>
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<tr>
<td>( f_8, x_0 = 1.5 )</td>
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<tr>
<td>( f_9, x_0 = 2 )</td>
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</table>

Form the table 1, we see that Algorithm 2.3 and algorithm 2.4 are comparable with other methods. It is clear from the table that for the examples \( f_6 \) and \( f_7 \), Newton method, methods of Corderro et al \[3\] and Weerakoon and Fernando \[12\] fail, while our methods find the approximate solutions. In view of this fact, one can consider our methods as an alternative to Newton method and the methods of \[3,12\] for solving nonlinear equations. Using the technique of this paper, one can obtain a cubic convergent method for solving a system of nonlinear equations and this is an interesting topic of future research.

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**References**