SOME NEW SIXTH ORDER VARIANTS OF NEWTON’S METHOD

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Abstract. In this paper, we present some new modifications of Newton’s method for solving non-linear equations. Analysis of convergence shows that these methods have order of convergence six. Numerical tests verifying the theory are given and based on these methods.

1. Introduction

Solving non-linear equations

\[ f(x) = 0, \]  
(1.1)

to find a simple root, where \( f : D \subset \mathbb{R} \rightarrow \mathbb{R} \) for an open interval \( D \) is a scalar function, is one of the most important problems in numerical analysis.

The classical Newton’s method for solving (1.1) is written as

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0. \]  
(1.2)

This is an important and basic method [4, 11], which converges quadratically.

Some modifications of Newton’s method with third-order of convergence have been developed in [2, 5] by considering different quadrature formulae for the computation of the integral arising from Newton’s theorem.

\[ f(x) = f(x_n) + \int_{x_n}^{x} f'(t) dt. \]  
(1.3)
Weerakoon and Fernando [13] rederived the classical Newton’s method by rectangular rule to compute the integral (1.3) and by the trapezoidal approximation, they arrive at an implicit scheme:

\[ x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_{n+1}) + f'(x_n)}, \quad (1.4) \]

which requires having the \((n + 1)\)th iterate \(x_{n+1}\) to calculate itself. They overcome this difficulty by making use of Newton’s iterative step to compute the \((n + 1)\)th iterate on the right-hand side of (1.4). So a modified Newton’s method with cubic convergence is obtained

\[ x_{n+1} = x_n - \frac{2f(x_n)}{f'(y_n) + f'(x_n)}, \quad (1.5) \]

where

\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0. \quad (1.6) \]

In [5], Homeier obtained the following mid-point method:

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(\frac{1}{2}(x_n + y_n))}. \quad (1.7) \]

In [12], Özban obtained a cubically convergent Newton’s type method as:

\[ x_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)} \left( \frac{1}{f'(x_n)} + \frac{1}{f'(y_n)} \right). \quad (1.8) \]

Kou et al. obtained the following method [8]:

\[ x_{n+1} = x_n - \frac{f(x_n)}{2} \left( \frac{1}{f'(x_n)} + \frac{1}{f'(\frac{1}{2}(x_n + y_n))} - f'(x_n) \right). \quad (1.9) \]

In (1.7), (1.8) and (1.9), \(y_n\) can be obtained from (1.6).

In this paper, we will study such improvements of the above modifications of Newton’s method. These methods are proved to have the sixth order of convergence. Their practical utility is demonstrated by numerical results.

2. Iterative methods and analysis of convergence

Here, we express the third order modifications of Newton’s method as a general form

\[ z_n = g_3(x_n). \quad (2.1) \]

Now, we consider the computation of the indefinite integral on a new interval of integration arising from Newton’s theorem

\[ f(x) = f(z_n) + \int_{z_n}^x f'(t)dt. \]

By the rectangular rule to compute the above integral, we obtain

\[ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad f'(z_n) \neq 0, \quad (2.2) \]
where \( z_n \) is defined by (2.1). We may use approximations by Taylor’s expansion as follows:

\[
f'(z_n) \approx f'(x_n) + f''(x_n)(z_n - x_n),
\]

where \( y_n \) is defined by (1.6).

From (2.3) and (2.4), we obtain

\[
f'(z_n) = \frac{f'(x_n)}{f(x_n)} (f(x_n) - (f'(y_n) - f'(x_n))(z_n - x_n)),
\]

and also a class of new methods as:

\[
x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n) - f'(x_n))(z_n - x_n)} f'(x_n),
\]

where \( z_n \) is defined by (2.1).

We now discuss the convergence analysis of the newly developed method (2.6).

**Theorem 2.1.** Assume that the function \( f: D \subset \mathbb{R} \to \mathbb{R} \) for an open interval \( D \) has a simple zero \( \alpha \in D \). Let \( f(x) \) is sufficiently differentiable in the interval \( D \), then the method defined by (2.6), in which \( z_n \) is defined by (2.1) and satisfies

\[
z_n - \alpha = A e_n^3 + O(e_n^4),
\]

for some \( A \neq 0 \), and \( e_n = x_n - \alpha \), has the order of convergence six.

**Proof.** Let \( \alpha \) be a simple zero of \( f \). By Taylor’s expansion, we have,

\[
f(x_n) = f'(\alpha) \left[ c_n + c_2 e_n^2 + \ldots \right],
\]

where

\[
c_k = \left( \frac{1}{k!} \right) \frac{f^{(k)}(\alpha)}{f'(\alpha)}, \quad k = 2, 3, \ldots, \text{ and } e_n = x_n - \alpha.
\]

From (2.8) and (2.9), we get

\[
\frac{f'(x_n)}{f'(x_n)} = c_n - c_2 e_n^2 + 2(c_2 - c_3) e_n^3 + (7c_2c_3 - 4c_2^2 - 3c_4) e_n^4 + \ldots,
\]

implies

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n^2 + 2(c_3 - c_2^2) e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3) e_n^4 + \ldots
\]

By Taylor’s series, we compute

\[
f'(y_n) = f'(\alpha) [c_2 e_n^2 + 2(c_2 - c_2^2) e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3) e_n^4 + \ldots],
\]

\[
f'(y_n) = f'(\alpha) \left[ 1 + 2c_2 e_n^2 + (4c_2c_3 - 4c_2^2) e_n^3 + (6c_2c_4 - 11c_2c_2^2) e_n^4 + \ldots \right],
\]

and

\[
f(z_n) = f'(\alpha) [(z_n - \alpha) + c_2 (z_n - \alpha)^2 + O((z_n - \alpha)^3)].
\]
Now from (2.6), (2.7), (2.8), (2.9), (2.13) and (2.14), we have

\[
e_{n+1} = z_n - \alpha - \frac{f(z_n)}{f(x_n) + f'(x_n)} f(x_n)
= z_n - \alpha - \left[ (z_n - \alpha) - c_2c_3e_n^3 z_n - \alpha + O(e_n^7) \right]
= c_2c_3(z_n - \alpha) c_n^3 + O(e_n^7)
= c_2c_3\alpha e_n^6 + O(e_n^7). \tag{2.15}
\]

Thus, we observe that the new method (2.6) has sixth order of convergence.

From (1.5), (1.6), (1.7), (1.8), (1.9) and (2.6), we can obtain new sixth order modifications of Newton’s method in which the corresponding value \( z_n \) is defined by (1.5), (1.7), (1.8) and (1.9) respectively:

\[
\begin{align*}
z_n &= x_n - \frac{2f(x_n)}{f'(y_n) + f'(x_n)}, \\
x_n+1 &= z_n - \frac{f(z_n)}{f(x_n) - (f'(y_n) - f'(x_n))(z_n - x_n)} f'(x_n), \tag{2.16} \\
z_n &= x_n - \frac{f(x_n)}{f'(\frac{1}{2}(x_n + y_n))}, \\
x_n+1 &= z_n - \frac{f(z_n)}{f(x_n) - (f'(y_n) - f'(x_n))(z_n - x_n)} f'(x_n), \tag{2.17} \\
z_n &= x_n - \frac{f(x_n)}{2 \left( \frac{1}{f'(x_n)} + \frac{1}{f'(y_n)} \right)}, \\
x_n+1 &= z_n - \frac{f(z_n)}{f(x_n) - (f'(y_n) - f'(x_n))(z_n - x_n)} f'(x_n), \tag{2.18} \\
\end{align*}
\]

and

\[
\begin{align*}
z_n &= x_n - \frac{f(x_n)}{2 \left( \frac{1}{f'(x_n)} + \frac{1}{f'(\frac{1}{2}(x_n + y_n))} - f'(x_n) \right)}, \\
x_n+1 &= z_n - \frac{f(z_n)}{f(x_n) - (f'(y_n) - f'(x_n))(z_n - x_n)} f'(x_n). \tag{2.19} \\
\end{align*}
\]

where

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)}, f'(x_n) \neq 0.
\]

3. Numerical examples

Iterative formulae (1.2), (1.5), (1.6), (1.7), (1.8) and (1.9) are respectively, called classical Newton’s method (NW), Weerakoon-Fernando method (WF), Homeier’s method (HM), Özban’s method (OZ) and Kou et al.’s method (KOU).

Here we use the logograms as, (FAW), (FAH), (FAO) and (FAK) to represent the present sixth-order methods defined by (2.16)-(2.19) respectively. The performance of the present methods with (NW), (WF), (HM), (OZ) and (KOU) is compared.
All the computations for above mentioned methods, are performed using software Maple 11, precision 60 digits and $\varepsilon = 10^{-15}$ as tolerance, using the following criteria for estimating the zero:

(i) $\delta = |x_{n+1} - x_n| < \varepsilon$,

(ii) $|f'(x_n)| < \varepsilon$,

(iii) Maximum numbers of iterations = 500.

We use following examples for numerical testing and the results are given in the Table-AF.

<table>
<thead>
<tr>
<th>Functions</th>
<th>Root</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$, $x_0 = -1$ arctan(x)</td>
<td>0</td>
</tr>
<tr>
<td>$f_2$, $x_0 = 3$ sin(x) e^{-x} + ln(x^2 + 1)</td>
<td>0.25753028543986</td>
</tr>
<tr>
<td>$f_3$, $x_0 = 3$ $(x - 1)^3 - 1$</td>
<td>2</td>
</tr>
<tr>
<td>$f_4$, $x_0 = 3$ sin(x) - x^2 + 1</td>
<td>1.40449164821534</td>
</tr>
<tr>
<td>$f_5$, $x_0 = 1.3$ $x^2 + 2 - x + 2 \cos(x) - 6$</td>
<td>1.82938360193384</td>
</tr>
</tbody>
</table>

### Table-AF

<table>
<thead>
<tr>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NW</td>
<td>-2.5e-83</td>
<td>-1.6e-74</td>
<td>7.1e-32</td>
<td>-2.5e-44</td>
</tr>
<tr>
<td>WF</td>
<td>3.5e-81</td>
<td>-7.0e-71</td>
<td>8.7e-56</td>
<td>-1.9e-44</td>
</tr>
<tr>
<td>HM</td>
<td>-1.2e-103</td>
<td>-7.6e-81</td>
<td>1.1e-62</td>
<td>-4.4e-70</td>
</tr>
<tr>
<td>OZ</td>
<td>3.4e-73</td>
<td>-2.4e-71</td>
<td>1.7e-101</td>
<td>1.2e-51</td>
</tr>
<tr>
<td>KOU</td>
<td>3.8e-47</td>
<td>-8.4e-70</td>
<td>-2.9e-48</td>
<td>9.6e-117</td>
</tr>
<tr>
<td>FAW</td>
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<td>1.6e-330</td>
<td>2.13e-66</td>
<td>-5.1e-48</td>
</tr>
<tr>
<td>FAH</td>
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<td>6.8e-147</td>
<td>1.7e-84</td>
<td>-7.8e-114</td>
</tr>
<tr>
<td>FAQ</td>
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<td>-1.4e-121</td>
<td>-8.6e-51</td>
<td>-1.5e-100</td>
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<tr>
<td>FAK</td>
<td>1.5e-287</td>
<td>1.5e-310</td>
<td>4.5e-113</td>
<td>1.0e-76</td>
</tr>
</tbody>
</table>

### 4. Conclusion

The results in Table-AF show that the new methods improve the computational efficiency of the previous methods, (NW), (WF), (HM), (OZ) and (KOU).

### References


