ON A GENERIC EXISTENCE RESULT FOR
A CLASS OF OPTIMIZATION PROBLEMS

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Abstract. In this work we study a minimization problem $f(x) \to \min, x \in C$ where $C$ is a subset of $X$ and $f : X \to \mathbb{R}^1$ is a continuous function defined on a complete metric space $X$ which is bounded from below on $C$. We assume that the set $C$ is the closure of its interior and show that for a generic function $f$ the minimization problem is well posed and its unique solution is an interior point of $C$.

1. Introduction

In this paper we consider a minimization problem

$$f(x) \to \min, x \in C$$

where $C$ is a nonempty closed subset of a complete metric space $X$ and $f : X \to \mathbb{R}^1$ belongs to a complete metric space of continuous functions defined on $X$ which are bounded from below on $C$. We assume that the set $C$ is the closure of its interior and show that for a generic function $f$ the minimization problem has a unique solution and this solution is an interior point of $C$.

When we say that a certain property holds for a generic element of a complete metric space $Y$ we mean that the set of points which have this property contains a $G_\delta$ everywhere dense subset of $Y$. Such an approach, when a certain property is investigated for the whole space $Y$ and not just for a single point in $Y$, has already been successfully applied in many areas of Analysis. We mention, for instance, the theory of dynamical systems [5, 8, 13, 15], optimization [9, 11, 12, 14], variational analysis [1], approximation theory [4, 6, 7], the calculus of variations [2, 3, 9, 16], and optimal control [17-19].

We use the convention that $\infty/\infty = 1$.

Let $(X, \rho)$ be a complete metric space and $C$ be a nonempty closed subset of $X$. For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{ y \in X : \rho(x, y) \leq r \}.$$
We assume that $C$ is the closure of its interior.

For each $f : X \to \mathbb{R}^1$ and each subset $K \subset X$ set

$$\inf(f; K) = \inf\{ f(x) : x \in K \}.$$ 

Denote by $\mathcal{M}$ the set of all continuous functions $f : X \to \mathbb{R}^1$ which are bounded from below on $C$. For each $f, g \in \mathcal{M}$ define

$$\tilde{d}(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$$ \hspace{1cm} (1.1)

and

$$d(f, g) = \tilde{d}(f, g)(1 + \tilde{d}(f, g))^{-1}.$$ 

It is not difficult to see that the metric space $(\mathcal{M}, d)$ is complete. Clearly, for each $f \in \mathcal{M}$, $\inf(f; C)$ is finite.

Let $f \in \mathcal{M}$. We say that the minimization problem $f(x) \to \min, x \in C$ is strongly well posed if $\inf(f; C)$ is attained at a unique point $x_f \in C$ and the following assertion holds:

For each $\epsilon > 0$ there exists $\delta > 0$ such that for each $g \in \mathcal{M}$ satisfying $d(f, g) \leq \delta$ and each $z \in C$ which satisfies $g(z) \leq \inf(g; C) + \delta$ the inequalities $\rho(x_f, z) \leq \epsilon$ and $|g(z) - f(x_f)| \leq \epsilon$ hold.

We will establish the following result.

**Theorem 1.1.** There exists a set $F \subset (\mathcal{M}, d)$ which is a countable intersection of open everywhere dense subsets of $(\mathcal{M}, d)$ such that for each $f \in \mathcal{M}$ the minimization problem $f(x) \to \min, x \in C$ is strongly well-posed and its unique minimizer is an interior point of $C$.

2. Basic lemma

For each $A \subset X$ denote by $\text{int}(A)$ the interior of $A$, by $\text{cl}(A)$ the closure of $A$ and by $\text{bd}(A)$ the set $\text{cl}(A) \setminus \text{int}(A)$.

For each $x \in X$ each $A \subset X$ set

$$\rho(x, A) = \inf(\rho(x, y) : y \in A).$$

We need the following well-known result [10, p. 115, p. 121].

**Proposition 2.1.** Let $A$ and $B$ be nonempty closed subsets of $(X, \rho)$ such that $A \cap B = \emptyset$. Then there exists a continuous function $h : X \to [0, 1]$ such that $h(x) = 1$ for all $x \in A$ and $h(x) = 0$ for all $x \in B$.

The following result is our basic lemma.

**Lemma 2.1.** Let $f \in \mathcal{M}$ and $\epsilon > 0$. Then there exist $f_* \in \mathcal{M}$ satisfying $d(f, f_*) \leq \epsilon$ and a neighborhood $\mathcal{U}$ of $f_*$ in $(\mathcal{M}, d)$ such that for each $g \in \mathcal{U}$

$$\inf(g; \text{int}(C)) < \inf(g; \text{bd}(C)).$$ \hspace{1cm} (2.1)
Proof. Choose a positive number
\[ \delta < 8^{-1} \min\{1, \epsilon\}. \] (2.2)
and choose \( x_0 \in C \) such that
\[ f(x_0) \leq \inf(f; C) + \delta/4. \] (2.3)
Since \( f \) is continuous there is a neighborhood \( V_0 \) of \( x_0 \) in \( (X, \rho) \) such that
\[ |f(x_0) - f(z)| \leq \delta/4 \text{ for all } z \in V. \] (2.4)
Since \( C \) is the closure of \( \text{int}(C) \) there is \( x^* \in X \) such that
\[ x^* \in V \cap \text{int}(C). \] (2.5)
(2.5) and (2.4) imply that
\[ |f(x_0) - f(x^*)| \leq \delta/4. \] (2.6)
(2.3) and (2.6) imply that
\[ f(x^*) \leq f(x_0) + \delta/4 \leq \inf(f; C) + \delta/2. \] (2.7)
Define \( f_1 : X \to \mathbb{R}^1 \) by
\[ f_1(x) = \max\{f(x), f(x^*)\} + 8^{-1} \epsilon \min\{1, \rho(x, x^*)\}, \quad x \in X. \] (2.8)
Clearly \( f_1 \in \mathcal{M} \) and for each \( x \in X \), \( f(x) \leq f_1(x) \). By (2.8), (2.7) and (2.2), for each \( x \in C \)
\[ 0 \leq f_1(x) - f(x) \leq 8^{-1} \epsilon + \max\{f(x), f(x^*)\} - f(x) \]
\[ \leq 8^{-1} \epsilon + \max\{f(x), \inf(f; C) + \delta/2\} - f(x) \]
\[ \leq 8^{-1} \epsilon + \delta/2 \]
\[ \leq 3 \cdot 16^{-1} \epsilon. \]
Thus
\[ 0 \leq f_1(x) - f(x) < 3 \cdot 16^{-1} \epsilon \text{ for all } x \in C. \] (2.9)
Since \( x^* \) is an interior point of \( C \) there is \( \gamma > 0 \) such that
\[ B(x^*, 4\gamma) \subset C. \] (2.10)
Therefore
\[ \rho(x^*, \text{bd}(C)) \geq 2\gamma. \] (2.11)
Set
\[ A = \{x \in C : \rho(x, \text{bd}(C)) \geq 2^{-1} \cdot 3\gamma\} \] (2.12)
and
\[ B = X \setminus \{x \in C : \rho(x, \text{bd}(C)) > \gamma\}. \] (2.13)
Clearly \( A \) is a closed subset of \( X \), \( B \) is a closed subset of \( X \) and
\[
A \cap B = \emptyset. \tag{2.14}
\]
By Proposition 2.1 there exists a continuous function \( \phi : X \to [0, 1] \) such that
\[
\phi(x) = 1, \ x \in A \text{ and } \phi(x) = 0, \ x \in B. \tag{2.15}
\]
Define
\[
f_*(x) = f_1(x) \phi(x) + (1 - \phi(x))(f(x) + \delta), \ x \in X. \tag{2.16}
\]
Clearly \( f_* \in \mathcal{M} \). By (2.9), (2.15) and (2.16) for each \( x \in C \),
\[
0 \leq f_*(x) - f(x) = \phi(x)(f_1(x) - f(x)) + (1 - \phi(x))\epsilon \leq \epsilon.
\]
If \( x \in X \setminus C \), then by (2.13) and (2.15)
\[
x \in B, \ \phi(x) = 0 \text{ and } f_*(x) = f(x) + \epsilon.
\]
Therefore
\[
0 \leq f_*(x) - f(x) \leq \epsilon \text{ for all } x \in X \text{ and } d(f, f_*) \leq \epsilon. \tag{2.17}
\]
We will show that for each \( x \in C \), \( f_*(x) \geq f_*(x_*) \). Let \( x \in C \). It follows from (2.16), (2.8), (2.7) and (2.2) that
\[
f_*(x) \geq \phi(x) \max\{f(x), f_*(x)\} + (1 - \phi(x))(f(x) + \epsilon)
\]
\[
\geq \phi(x)f(x_*) + (1 - \phi(x))(f(x) + \epsilon)
\]
\[
\geq \phi(x)f(x_*) + (1 - \phi(x))(\inf(f; C) + \epsilon)
\]
\[
\geq f(x_*)
\]
and
\[
f_*(x) \geq f(x_*) \tag{2.18}
\]
for all \( x \in C \). By (2.11), (2.12), (2.15), (2.16) and (2.8)
\[
x_* \in A, \ \phi(x_*) = 1 \text{ and } f_*(x_*) = f_1(x_*) = f(x_*). \tag{2.19}
\]
(2.19) and (2.18) imply that
\[
f_*(x) \geq f(x_*) = f_*(x_*) \text{ for all } x \in C. \tag{2.20}
\]
By (2.13), (2.15), (2.16), (2.7) and (2.2) for all \( x \in \text{bd}(C) \),
\[
x \in B, \ \phi(x) = 0
\]
and
\[
f_*(x) = f(x) + \epsilon \geq \inf(f; C) + \epsilon > f(x_*) + \delta/2.
\]
Combined with (2.19) and (2.20) this inequality implies that
\[ \inf(f_*; \text{bd}(C)) \geq f(x_*) + \delta/2 = f_*(x_*) + \delta/2 = \inf(f_*; C) + \delta/2. \]  

(2.21)

Define
\[ U = \{ g \in M : \tilde{d}(g, f_*) \leq \delta/8 \}. \]

(2.22)

Clearly for each \( g \in U \),
\[ |f_*(z) - g(z)| \leq \delta/8, \; z \in X, \]
\[ |\inf(f_*; C) - \inf(g; C)| \leq \delta/8, \]
\[ |\inf(f_*; \text{bd}(C)) - \inf(g, \text{bd}(C))| \leq \delta/8 \]

and
\[ \inf(g; \text{bd}(C)) \geq -\delta/8 + \inf(f_*; \text{bd}(C)) \]
\[ \geq -\delta/8 + \inf(f_*; C) + \delta/2 \]
\[ \geq \delta/2 - \delta/8 + \inf(g; C) - \delta/8 \]
\[ \geq \inf(g; C) + \delta/4. \]

Thus
\[ \inf(g; \text{bd}(C)) \geq \inf(g; C) + \delta/4 \text{ for all } g \in U. \]

Lemma 2.1 is proved.

Lemma 2.1 implies the following result.

**Proposition 2.2.** There exists an everywhere dense open subset \( F_1 \) of \((M, d)\) such that for each \( f \in F_1 \),
\[ \inf(f; C) < \inf(f; \text{bd}(C)). \]

**Proof.** By Lemma 2.1 for each \( f \in M \) and each integer \( i \geq 1 \) there exists a nonempty open set \( U(f, i) \subset M \) such that
\[ U(f, i) \cap \{ g \in M : d(f, g) \leq i^{-1} \} \neq \emptyset \]

and
\[ \inf(g; C) < \inf(g; \text{bd}(C)) \text{ for each } g \in U(f, i). \]

Set
\[ F_1 = \bigcup \{ U(f, i) : f \in M, \; i = 1, 2, \ldots \}. \]

Evidently \( F_1 \) is an open everywhere dense subset of \( M \). It is easy to see that for each \( g \in F_1 \)
\[ \inf(g; C) < \inf(g; \text{bd}(C)). \]

Proposition 2.2 is proved.
3. Auxiliary result

**Proposition 3.1.** There exists a set $\mathcal{F}_0 \subset \mathcal{M}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}$ such that for each $f \in \mathcal{F}_0$ the minimization problem $f(x) \rightarrow \min_x$, $x \in C$ is strongly well-posed.

We obtain Proposition 3.1 as a realization of the variational principle (see [9, 17]). By Theorem 2.2 of [9], Proposition 3.1 follows from the next lemma.

**Lemma 3.1.** For any $f \in \mathcal{M}$, any $\epsilon > 0$ and any $\gamma > 0$ there exists a nonempty open set $W$ in $\mathcal{M}$, $x \in C$, $\alpha \in \mathbb{R}$, and $\eta > 0$ such that for each $h \in W$

$$d(f, h) < \epsilon$$

and for each $h \in W$, if $z \in C$ is such that $h(z) \leq \inf(h; C) + \eta$, then

$$\rho(x, z) \leq \gamma \quad \text{and} \quad |h(z) - \alpha| \leq \gamma.$$

**Proof.** Let $f \in \mathcal{M}$, $\epsilon > 0$ and $\gamma > 0$. Choose a positive number

$$\eta < 64^{-1} \min\{1, \epsilon\} \min\{1, \gamma\}, \quad (3.1)$$

choose a point $\bar{x} \in C$ such that

$$f(\bar{x}) \leq \inf(f; C) + \eta \quad (3.2)$$

and define

$$\bar{f}(z) = f(z) + 8^{-1} \epsilon \min\{1, \rho(z, \bar{x})\}, \quad z \in X. \quad (3.3)$$

Clearly $\bar{f} \in \mathcal{M}$. Denote by $W$ an open neighborhood of $\bar{f}$ in $\mathcal{M}$ such that

$$W \subset \{h \in \mathcal{M} : \tilde{d}(\bar{f}, h) \leq \eta/8\}. \quad (3.4)$$

(3.3) implies that

$$d(f, \bar{f}) \leq \tilde{d}(\bar{f}, f) \leq 8^{-1} \epsilon.$$  

Combined with (3.4) this implies that for each $h \in W$,

$$d(f, h) \leq \tilde{d}(f, h) \leq \tilde{d}(f, \bar{f}) + \tilde{d}(\bar{f}, h) \leq \epsilon/8 + \eta/8 < \epsilon. \quad (3.5)$$

Let $h \in W$. It follows from (3.4) that

$$|\bar{f}(z) - h(z)| < \eta \quad \text{for all} \quad z \in X. \quad (3.6)$$

(3.6) implies that

$$|\inf(\bar{f}; C) - \inf(h; C)| \leq \eta. \quad (3.7)$$

Assume that $z \in C$ and

$$h(z) \leq \inf(h; C) + \eta. \quad (3.8)$$
By this inequality, (3.3), (3.6), (3.7), (3.8), (3.2) and (3.1),
\[
\begin{align*}
f(z) + 8^{-1} \epsilon \min \{1, \rho(z, \bar{x})\} &= \bar{f}(z) \\
&\leq h(z) + \eta \\
&\leq \inf(h; C) + 2\eta \\
&\leq \inf(\bar{f}; C) + 3\eta \\
&\leq \bar{f}(\bar{x}) + 3\eta \\
&= f(\bar{x}) + 3\eta \\
&\leq f(z) + 4\eta
\end{align*}
\]
and
\[
\min \{1, \rho(z, \bar{x})\} \leq 32\eta^{-1} \leq \min \{1, \gamma\}/2.
\]
Therefore
\[
\rho(z, \bar{x}) \leq \gamma/2. \tag{3.9}
\]
It follows from (3.8), (3.7) and (3.1) that
\[
|h(z) - \inf(\bar{f}; C)| \leq |h(z) - \inf(h; C)| + |\inf(h; C) - \inf(\bar{f}; C)| \leq 2\eta \leq \gamma
\]
and $|h(z) - \inf(\bar{f}; C)| \leq \gamma$. Combined with (3.9) this inequality implies that
\[
|h(z) - \inf(\bar{f}; C)| \leq \gamma, \quad \rho(z, \bar{x}) \leq \gamma/2.
\]
This completes the proof of Lemma 3.1.

4. Proof of Theorem 1.1

By Proposition 2.2 there exists an everywhere dense open subset $F_1$ of $(\mathcal{M}, d)$ such that for each $f \in F_1$,
\[
\inf(f; C) < \inf(f; \text{bd}(C)). \tag{4.1}
\]
By Proposition 3.1 there exists a set $F_0 \subset \mathcal{M}$ which is a countable intersection of open everywhere dense subsets of $(\mathcal{M}, d)$ such that for each $f \in F_0$ the minimization problem $f(x) \to \min$, $x \in C$ is strongly well-posed. Set
\[
F = F_0 \cap F_1.
\]
Clearly $F$ is a countable intersection of open everywhere dense subsets of $(\mathcal{M}, d)$.

Let $f \in F$. Then (4.1) holds and the minimization problem $f(x) \to \min$, $x \in C$ is strongly well-posed. Denote by $x_f$ its unique solution. Then by (4.1) $x_f$ is an interior point of $C$. Theorem 1.1 is proved.
References


