A NEW PROXIMAL POINT ALGORITHM WITH ERRORS FOR NONLINEAR VARIATIONAL INCLUSIONS INVOLVING GENERALIZED $m$-ACCRETIVE MAPPINGS

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Abstract. In this paper, we introduce and study a new class of nonlinear variational inclusions involving generalized $m$-accretive mappings and construct a new proximal point algorithm with errors for solving this class of nonlinear variational inclusions by using the resolvent operator technique for generalized $m$-accretive mapping due to Huang and Fang. Those results improve and generalize some recent results in this field.

1. Introduction

In recent years, variational inequalities have been generalized and extended in many different directions using novel and innovative techniques to study wider classes of unrelated problems arising in optimization and control, economic and finance, transportation and electrical networks, operations research and engineering sciences in a general and unified framework, see [1-14, 16-18, 20-25, 27, 28] and the references therein. Useful and important generalizations of variational inequalities are variational inclusions, which have been studied by Adly [1], Ding [4, 5], Hassouni and Moudufi [7], Huang [8, 10, 11], Huang and Fang [14], Noor [20] and Uko [25], and many others in the Hilbert spaces settings (see, for example, [27] and the references therein).

Recently, Huang and Fang [15] introduced the concept of generalized $m$-accretive mapping, which is a generalization of the $m$-accretive mapping, and studied the properties of the resolvent operator associated with the generalized $m$-accretive mapping in Banach spaces. Furthermore, Huang [11], Huang-Fang-Deng [16], and Jin [17] introduced and studied some new classes of nonlinear variational inclusions involving generalized $m$-accretive mappings in Banach spaces. By using the resolvent operator technique in [15], they constructed some iterative algorithms for solving the nonlinear variational inclusions involving generalized $m$-accretive mappings. They also proved the existence of solution for nonlinear variational inclusions involving generalized $m$-accretive mappings and convergence of sequences generated by the algorithms.

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Motivated and inspired by recent research works in this field, in this paper, we introduce and study a new class of nonlinear variational inclusions involving generalized $m$-accretive mappings and construct a new proximal point algorithm with errors for solving this class of nonlinear variational inclusions by using the resolvent operator technique for generalized $m$-accretive mapping due to Huang and Fang [15]. Our results improve and generalize the corresponding results of [5, 8-14, 16-18, 23, 24, 27].

2. Preliminaries

Let $X$ be a real Banach space with dual space $X^*$, $\langle\cdot,\cdot\rangle$ be the dual pair between $X$ and $X^*$. Let $2^X$, $CB(X)$, and $H(\cdot,\cdot)$ denote the family of all the nonempty subsets of $X$, the family of all the nonempty closed bounded subsets of $X$, and the Hausdorff metric on $CB(X)$, respectively. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|f^*\|\|x\|\text{ and } \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, $J_2$ is the usual normalized duality mapping. It is known that, $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \neq 0$ and $J_q$ is single-valued if $X^*$ is strictly convex (see, for example, [26]). If $X = H$ is a Hilbert space, then $J_2$ becomes the identity mapping of $H$. In the sequel we shall denote the single-valued generalized duality mapping by $j_q$.

**Definition 2.1.** A mapping $g : X \to X$ is said to be

(i) accretive if for any $u, v \in X$, there exists $j_q(u - v) \in J_q(u - v)$ such that

$$\langle g(u) - g(v), j_q(u - v) \rangle \geq 0;$$

(ii) $\alpha$-strongly accretive for any $u, v \in X$, there exists $j_q(u - v) \in J_q(u - v)$ such that

$$\langle g(u) - g(v), j_q(u - v) \rangle \geq \alpha\|u - v\|^q,$$

where $\alpha > 0$ is a constant.

(iii) $m$-accretive if $g$ is accretive and $(I + \lambda g)(X) = X$ for all (equivalently, for some) $\lambda > 0$, where $I$ denotes the identity mapping.

(iv) $\beta$-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|g(u) - g(v)\| \leq \beta\|u - v\|, \quad \forall u, v \in X.$$

**Definition 2.2.** A set-valued mapping $S : X \to CB(X)$ is said to be $H$-Lipschitz continuous if there exists a constant $\sigma > 0$ such that $H(Su, Sv) \leq \sigma\|u - v\|, \quad \forall u, v \in X$. 


Definition 2.3. Let \( p : X \to X \) be a mapping. A mapping \( N : X \times X \to X \) is said to be
(i) \( \gamma \)-strongly accretive with respect to \( p \) in the first argument if there exists a constant \( \gamma > 0 \) and for any \( u, v \in X \) there exists \( j_q(u-v) \in J_q(u-v) \) such that
\[
\langle N(p(u), \cdot) - N(p(v), \cdot), j_q(u-v) \rangle \geq \gamma \| u - v \|^q;
\]
(ii) Lipschitz continuous in the first argument if, there exists a constant \( \mu > 0 \) such that
\[
\| N(u, \cdot) - N(v, \cdot) \| \leq \mu \| u - v \|, \quad \forall u, v \in X.
\]
Similarly, we can define the Lipschitz continuity of \( N(\cdot, \cdot) \) in the second argument.

Definition 2.4. A mapping \( \eta : X \times X \to X^* \) is said to be
(i) monotone if \( \langle u - v, \eta(u, v) \rangle \geq 0 \), \( \forall u, v \in X \);
(ii) strictly monotone if \( \langle u - v, \eta(u, v) \rangle \geq 0 \), \( \forall u, v \in X \) and equality holds if and only if \( u = v \);
(iii) strongly monotone if there exists a constant \( \delta > 0 \) such that
\[
\langle u - v, \eta(u, v) \rangle \geq \delta \| u - v \|^2, \quad \forall u, v \in X;
\]
(iv) Lipschitz continuous if there exists a constant \( \tau > 0 \) such that
\[
\| \eta(u, v) \| \leq \tau \| u - v \|, \quad \forall u, v \in X.
\]

Definition 2.5 [15]. Let \( \eta : X \times X \to X^* \) be a single-valued mapping. A set-valued mapping \( M : X \to 2^X \) is said to be
(i) \( \eta \)-accretive if \( \langle x - y, \eta(u, v) \rangle \geq 0 \), \( \forall u, v \in X, x \in Mu, y \in Mv \);
(ii) generalized \( m \)-accretive if \( M \) is \( \eta \)-accretive and \( (I + \lambda M)(X) = X \) for any \( \lambda > 0 \).

Remark 2.1. Huang and Fang gave one example of generalized \( m \)-accretive mapping in [15].

Lemma 2.1 [15]. Let \( \eta : X \times X \to X^* \) be strictly monotone and \( M : X \to 2^X \) be a generalized \( m \)-accretive mapping. Then the following conclusions hold:
(1) \( \langle x - y, \eta(u, v) \rangle \geq 0 \) for all \( (v, y) \in Graph(M) \) implies that \( (u, x) \in Graph(M) \), where \( Graph(M) = \{(u, x) \in X \times X : x \in Mu\} \);
(2) the inverse mapping \( (I + \lambda M)^{-1} \) is single-valued for any \( \lambda > 0 \).

Based on Lemma 2.1, we can define the resolvent operator for a generalized \( m \)-accretive mapping as follows:
\[
J^M_\rho(z) = (I + \rho M)^{-1}(z) \quad \text{for all } z \in X,
\]
where \( \rho > 0 \) is a constant and \( \eta : X \times X \to X^* \) is a strictly monotone mapping.
Lemma 2.2 [15]. Let $\eta : X \times X \to X^*$ be strongly monotone and Lipschitz continuous with constants $\delta > 0$ and $\tau > 0$, respectively. Let $M : X \to 2^X$ be a generalized $m$-accretive mapping. Then the resolvent operator $J^M_\rho$ for $M$ is Lipschitz continuous with constant $\tau/\delta$, i.e.,

$$
\|J^M_\rho(u) - J^M_\rho(v)\| \leq \frac{\tau}{\delta}\|u - v\| \text{ for all } u, v \in X.
$$

Lemma 2.3 [26]. Let $X$ be a real uniformly smooth Banach space. Then $X$ is $q$-uniformly smooth if and only if there exists a constant $C_q > 0$ such that for all $x, y \in X$,

$$
\|x + y\|^q \leq \|x\|^q + q(g, j_q(x)) + C_q\|y\|^q.
$$

Let $\eta : X \times X \to X^*$ and $N : X \times X \to X$ be two single-valued mappings with two variables, and $f, g, p : X \to X$ be three single-valued mappings. Let $S, T, G : X \to CB(X)$ be three set-valued mappings and $M : X \times X \to 2^X$ be a set-valued mapping such that for each $t \in X$, $M(\cdot, t)$ is generalized $m$-accretive mapping with $\text{Range}(g) \cap \text{Dom}(M(\cdot, t)) \neq \emptyset$. Now we consider the following problem:

Find $u \in X, x \in Su$ and $y \in Tu$ and $w \in Gu$ such that

$$
0 \in f(x) + N(p(u), y) + M(g(u), w). 
$$

(2.1)

Problem (2.1) is called the nonlinear variational inclusion.

If $X = X^* = H$ is a Hilbert space, Then the following some special cases of problem (2.1):

(I) If $M(x, t) = M(x)$ for all $x, t$ in $H$, then problem (2.1) reduces to the following problem:

Find $u \in H, x \in Su$ and $y \in Tu$ such that

$$
0 \in f(x) + N(p(u), y) + M(g(u)),
$$

(2.2)

which appears to be a new one. Furthermore, if $T$ is a single-valued mapping, $f = 0$, and $\eta(x, y) = x - y$ for all $x, y \in H$, then problem (2.2) is equivalent to the variational inclusion considered by Huang [10].

(II) If $M(\cdot, t) = \Delta \phi(\cdot, t)$, where $\phi : H \times H \to \mathbb{R}\cup\{\infty\}$ is a functional such that for each fixed $t$ in $H$, $\phi(\cdot, t) : H \to \mathbb{R}\cup\{\infty\}$ is lower semicontinuous and $\eta$-subdifferentiable on $H$, and $\Delta \phi(\cdot, t)$ denotes the $\eta$-subdifferential of $\phi(\cdot, t)$, then problem (2.1) reduces to the following problem:

Find $u \in H, x \in Su, y \in Tu$ and $z \in Gu$ such that

$$
(f(x) + N(p(u), y), \eta(v, g(u))) \succeq \phi(g(u), z) - \phi(v, z)
$$

(2.3)

for all $v$ in $H$, which appears to be a new one. Furthermore, if $f = 0$, $N(x, y) = x - y$ for all $x, y$ in $H$, and $T$ is a single-valued mapping, then problem (2.3) reduces to the general quasi-variational-like inclusion considered by Ding and Luo [5].

(III) If $T : H \to H$ is a single-valued mapping, $g$ is an identity mapping, $f = 0$, $N(x, y) = x - y$ for all $x, y$ in $H$, and $M(\cdot, t) = \Delta \phi$ for all $t$ in $H$, where $\Delta \phi$ denotes
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the $\eta$-subdifferential of a proper convex lower semicontinuous function $\phi : H \to \mathbb{R} \cup \{+\infty\}$, then problem (2.1) reduces to the following problem:

Find $u \in H$ such that

$$
\langle p(u) - Tu, \eta(v, u) \rangle \geq \phi(u) - \phi(v) \quad (2.4)
$$

for all $v$ in $H$, which is called the strongly nonlinear variational-like inclusion problem considered by Lee et al. [18].

(IV) If $\eta(x, y) = x - y$ and $M(\cdot, t) = \partial \phi$, where $\partial \phi$ denotes the subdifferential of a proper convex lower semicontinuous function $\phi : H \to \mathbb{R} \cup \{+\infty\}$, then problem (2.1) reduces to finding $u \in H, x \in Su$ and $y \in Tu$ such that

$$
\langle f(x) + N(p(u), y), v - g(u) \rangle \geq \phi(g(u)) - \phi(v) \quad (2.5)
$$

for all $v$ in $H$. Furthermore, if $f = 0$, $N(x, y) = x - y$ for all $x, y$ in $H$ and $g$ is an identity mapping, then problem (2.5) is equivalent to the set-valued nonlinear generalized variational inclusion considered by Huang [8].

For a suitable choice of $N, \eta, f, g, p, S, T, G, M$, and the space $X$, one can obtained a number of known and new classes of variational inclusions, variational inequalities, and corresponding optimization problems from nonlinear variational inclusion problem (2.1). Furthermore, these types of variational inclusions enable us to study many important problems arising in the mathematical, physical, and engineering sciences in a general and unified framework.

3. Main results

Lemma 3.1. For given $u \in X, x \in Su, y \in Tu$ and $w \in Gu$, $(u, x, y, w)$ is a solution of problem (2.1) if and only if

$$
g(u) = J^M_{\rho}(\cdot, w)(g(u)) - \rho(f(x) + N(p(u), y)), \quad (3.1)
$$

where $J^M_{\rho}(\cdot, w) = (I + \rho M(\cdot, w))^{-1}$ and $\rho > 0$ is a constant.

Proof. This directly follows from the definition of $J^M_{\rho}(\cdot, w)$.

Remark 3.1. From Lemma 3.1, we know that problem (2.1) is equivalent to fixed point problem (3.1). Equation (3.1) can be written as

$$
u = u - g(u) + J^M_{\rho}(\cdot, w)(g(u) - \rho(f(x) + N(p(u), y))). \quad (3.2)
$$

This fixed point formulation enables us to suggest the following proximal point algorithm with errors.
Algorithm 3.1. For any given $u_0 \in X$, $x_0 \in Su_0$, $y_0 \in Tu_0$, and $w_0 \in Gu_0$. Let
\[
u_1 = (1 - \lambda)u_0 + \lambda[u_0 - g(u_0) + J^M_{\rho}(\cdot, \omega_0)(g(u_0) - \rho(f(x_0) + N(p(u_0), y_0)))] + \lambda e_0,
\]
where $\rho > 0$ and $0 < \lambda < 1$ are constants, $e_0 \in X$.

Since $Su_0, Su_1 \in CB(X)$, $Tu_0, Tu_1 \in CB(X)$ and $Gu_0, Gu_1 \in CB(X)$, it follows from Nadler [19] that there exist $x_1 \in Su_1$, $y_1 \in Tu_1$ and $w_1 \in Gu_1$ such that
\[
\begin{align*}
\|x_0 - x_1\| & \leq (1 + 1)H(Su_0, Su_1), \\
\|y_0 - y_1\| & \leq (1 + 1)H(Tu_0, Tu_1), \\
\|w_0 - w_1\| & \leq (1 + 1)H(Gu_0, Gu_1).
\end{align*}
\]

Let
\[
u_2 = (1 - \lambda)u_1 + \lambda[u_1 - g(u_1) + J^M_{\rho}(\cdot, \omega_1)(g(u_1) - \rho(f(x_1) + N(p(u_1), y_1)))] + \lambda e_1,
\]
where $\rho > 0$ and $0 < \lambda < 1$ are constants, $e_1 \in X$.

Continuing this way, we can define sequences $\{u_n\}, \{x_n\}, \{y_n\}$ and $\{w_n\}$ as follows: for all $n \geq 0$,
\[
\begin{align*}
\left\{ \begin{array}{l}
\nu_{n+1} = (1 - \lambda)u_n \\
\quad + \lambda[u_n - g(u_n) + J^M_{\rho}(\cdot, \omega_n)(g(u_n) - \rho(f(x_n) + N(p(u_n), y_n)))] \\
\quad + \lambda e_n, \\
x_n \in Su_n, \ |x_n - x_{n+1}| \leq (1 + (1 + n)^{-1})H(Su_n, Su_{n+1}), \\
y_n \in Tu_n, \ |y_n - y_{n+1}| \leq (1 + (1 + n)^{-1})H(Tu_n, Tu_{n+1}), \\
w_n \in Gu_n, \ |w_n - w_{n+1}| \leq (1 + (1 + n)^{-1})H(Gu_n, Gu_{n+1}),
\end{array} \right. \tag{3.3}
\end{align*}
\]

where $\rho > 0$ and $0 < \lambda < 1$ are constants, $\{e_n\} \subset X$ is an error sequence which is taken into account a possible inexact computation of the proximal point.

Now we show the existence of solutions of problem (2.1) and convergence of sequences generated by Algorithm 3.1.

Theorem 3.1. Let $X$ be a real $q$-uniformly smooth Banach space and $\eta : X \times X^* \to X^*$ be strongly monotone and Lipschitz continuous with constants $\delta$ and $\tau$, respectively. Let $S, T, G : X \to CB(X)$ be $H$-Lipschitz continuous with constants $\sigma, \kappa, \iota$, respectively, and $g : X \to X$ be $\alpha$-strongly accretive and $\beta$-Lipschitz continuous. Let $f : X \to X$ be $r$-Lipschitz continuous and $N : X \times X \to X$ be Lipschitz continuous in the first and second arguments with constants $\mu$ and $\nu$, respectively. Let $p : X \to X$ be $\gamma$-strongly accretive with respect to the first argument of $N$ and $\xi$-Lipschitz continuous. Let $M : X \times X \to 2^X$ be a set-valued mapping such that, for each fixed $t \in X$, $M(\cdot, t)$ is a generalized $m$-accretive mapping and $\text{Range}(g) \cap \text{Dom}M(\cdot, t) \neq \emptyset$. Suppose that there exist constants $\rho > 0$ and $\zeta > 0$ such that for each $x, y, z \in X$,
\[
\|J^M_{\rho}(\cdot, x)(z) - J^M_{\rho}(\cdot, y)(z)\| \leq \zeta \|x - y\|, \tag{3.4}
\]
Then the iterative sequences \{u_n\}, \{x_n\}, \{y_n\} and \{w_n\} generated by Algorithm 3.1 strongly converge to \(u^*, x^*, y^*\) and \(w^*\), respectively, and \((u^*, x^*, y^*, w^*)\) is a solution of problem (3.1).

**Proof.** From Algorithm 3.1, Lemma 2.2 and (3.4), we have

\[
\begin{aligned}
&\|u_{n+1} - u_n\| \\
\leq & (1 - \lambda)\|u_n - u_{n-1}\| + \lambda\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| + \lambda\|e_n - e_{n-1}\| \\
&+ \|J_{\rho}^M(u_n) - \rho(f(x_n) + N(p(u_n), y_n))\| \\
&+ \|J_{\rho}^M(u_{n-1}) - \rho(f(x_{n-1}) + N(p(u_{n-1}), y_{n-1})))\| \\
\leq & (1 - \lambda)\|u_n - u_{n-1}\| + \lambda\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| + \lambda\|e_n - e_{n-1}\| \\
&+ \lambda\|J_{\rho}^M(u_n) - \rho(f(x_n) + N(p(u_n), y_n))\| \\
&+ \lambda\|J_{\rho}^M(u_{n-1}) - \rho(f(x_{n-1}) + N(p(u_{n-1}), y_{n-1})))\| \\
\leq & (1 - \lambda)\|u_n - u_{n-1}\| + \lambda\|e_n - e_{n-1}\| + \lambda\|w_n - w_{n-1}\| \\
&+ \lambda(1 + \frac{T}{\delta})\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| \\
&+ \lambda(1 + \frac{T}{\delta})\|u_n - u_{n-1} - \rho(N(p(u_n), y_n) - N(p(u_{n-1}), y_{n-1})))\| \\
&+ \lambda\|\rho\|N(p(u_{n-1}), y_n) - N(p(u_{n}-1), y_{n-1}))\| + \|f(x_n) - f(x_{n-1})\|
\end{aligned}
\]  

(3.7)

Since \(g\) is \(\alpha\)-strongly accretive and \(\beta\)-Lipschitz continuous,

\[
\begin{aligned}
&\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\|^q \\
\leq & \|u_n - u_{n-1}\|^q - q\|g(u_n) - g(u_{n-1}), j_q(u_n - u_{n-1})\| \\
&+ C_q\|g(u_n) - g(u_{n-1})\|^q \\
\leq & (1 - \alpha q + C_q\beta^q)\|u_n - u_{n-1}\|^q.
\end{aligned}
\]  

(3.8)
Further, from the assumptions, we get

\[
\|u_n - u_{n-1} - \rho(N(p(u_n), y_n) - N(p(u_{n-1}), y_n))\|^q \\
\leq \|u_n - u_{n-1}\|^q - q\rho(N(p(u_n), y_n) - N(p(u_{n-1}, y_n), j_q(u_n - u_{n-1})) \\
+ C_q\rho\|N(p(u_n), y_n) - N(p(u_{n-1}), y_n)\|^q \\
\leq (1 - q\rho\gamma)\|u_n - u_{n-1}\|^q + C_q\rho^q\|p(u_n) - p(u_{n-1})\|^q \\
\leq (1 - q\rho\gamma + C_q\rho^q\gamma)\|u_n - u_{n-1}\|^q, 
\]

(3.9)

\[
\|N(p(u_n), y_n) - N(p(u_{n-1}, y_n))\| \leq \nu\|y_n - y_{n-1}\| 
\]

(3.10)

\[
\|f(x_n) - f(x_{n-1})\| \leq r\|x_n - x_{n-1}\| \leq r\sigma(1 + \frac{1}{1 + n})\|u_n - u_{n-1}\|. 
\]

(3.11)

It follows from (3.7)-(3.11) that

\[
\|u_{n+1} - u_n\| \leq (1 - \lambda + \lambda\theta_n)\|u_n - u_{n-1}\| + \lambda\|e_n - e_{n-1}\|, 
\]

(3.12)

where

\[
\theta_n = (1 + \frac{r}{\delta})(1 - q\alpha + C_q\beta)^{\frac{1}{2}} + \frac{r}{\delta}(1 - q\rho\gamma + C_q\rho^q\gamma)^{\frac{1}{2}} \\
+ \frac{r}{\delta}\nu\kappa + r\sigma)(1 + \frac{1}{1 + n}) + \zeta(1 + \frac{1}{1 + n}). 
\]

It follows from (3.5) that \(\theta_n \to \theta\) as \(n \to \infty\), and so there exist a number \(N\), such that \(\theta_n < h\) for all \(n \geq N\). Therefore for all \(n \geq N\), (3.12) reduces to

\[
\|u_{n+1} - u_n\| \leq (1 - \lambda + \lambda h)\|u_n - u_{n-1}\| + \lambda\|e_n - e_{n-1}\| \\
\leq \epsilon^n\|u_1 - u_0\| + \sum_{k=1}^{n} \epsilon^{n-k}\lambda\|e_k - e_{k-1}\|, 
\]

where \(\epsilon = 1 - \lambda + \lambda h\). For any \(m > n > 0\), we have

\[
\|u_m - u_n\| \leq \sum_{j=n}^{m-1} \|u_{j+1} - u_j\| \leq \sum_{j=n}^{m-1} \epsilon^j\|u_1 - u_0\| + \sum_{j=n}^{m-1} \sum_{k=1}^{j} \epsilon^{j-k}\lambda\|e_k - e_{k-1}\|. 
\]

By (3.5) and (3.6), we have \(\lim_{m,n \to \infty} \|x_m - x_n\| = 0\), and hence \(\{u_n\}\) is a Cauchy sequence in \(X\). Let \(x_n \to x^*\) as \(n \to \infty\). From (3.3), we get

\[
\|x_n - x_{n+1}\| \leq (1 + \frac{1}{1 + n})H(Su_n, Su_{n+1}) \\
\leq \sigma(1 + \frac{1}{1 + n})\|u_n - u_{n+1}\|. 
\]

(3.13)
\[ \|y_n - y_{n+1}\| \leq (1 + \frac{1}{1+n})H(Tu_n, Tu_{n+1}) \]
\[ \leq \kappa(1 + \frac{1}{1+n})\|u_n - u_{n+1}\|. \]  
(3.14)

\[ \|w_n - w_{n+1}\| \leq (1 + \frac{1}{1+n})H(Gu_n, Gu_{n+1}) \]
\[ \leq \iota(1 + \frac{1}{1+n})\|u_n - u_{n+1}\|. \]  
(3.15)

Since \(\{u_n\}\) is a Cauchy sequence, (3.13), (3.14) and (3.15) imply that \(\{x_n\}, \{y_n\}\) and \(\{w_n\}\) are also Cauchy sequences. Let \(x_n \rightarrow x^*, y_n \rightarrow y^*\) and \(w_n \rightarrow w^*\) as \(n \rightarrow \infty\).

Furthermore, we have

\[ d(x^*, Su^*) \leq \|x^* - x_n\| + d(x_n, Su^*) \]
\[ \leq \|x^* - x_n\| + H(Su_n, Su^*) \]
\[ \leq \|x^* - x_n\| + \sigma\|u_n - u^*\| \rightarrow 0. \]

It follows that \(x^* \in Su^*\). Similarly, we know that \(y^* \in Tu^*\) and \(w^* \in Gu^*\).

By the assumptions, Algorithm 3.1 and \(\lim_{k \rightarrow \infty} \|e_k\| = 0\), we have

\[ u^* = (1 - \lambda)u^* + \lambda(u^* - g(u^*) + J_M^M(w^*)(g(u^*) - \rho(f(x^*) + N(p(u^*), y^*)'))) \]

i. e.,
\[ g(u^*) = J_M^M(w^*)(g(u^*) - \rho(f(x^*) + N(p(u^*), y^*)')). \]

By Lemma 3.1, we know that \((u^*, x^*, y^*, w^*)\) is a solution of problem (2.1).

This completes the proof.

Remark 3.2. Theorems 3.1 extends and improves some corresponding results of [5, 8-16, 14-18, 23, 24, 27].

References

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