SENSITIVITY ANALYSIS FOR STRONGLY NONLINEAR QUASI-VARIATIONAL INCLUSIONS INVOLVING GENERALIZED $m$-ACCRETIVE MAPPINGS

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Abstract. In this paper, we introduce and study a new class of strongly nonlinear quasi-variational inclusions involving generalized $m$-accretive mapping in Banach spaces. By using the resolvent operator technique for generalized $m$-accretive mapping due to Huang and Fang, we prove the existence and uniqueness of solutions, and study the sensitivity analysis for this class of strongly nonlinear quasi-variational inclusions involving generalized $m$-accretive mapping in Banach spaces. Our results improve and generalize some known results in the literature.

1. Introduction

Recently, in order to study a wide of problems arising in industry, physical, regional, social, pure and applied sciences, the classical variational inequality problems have been extended and generalized in many directions. The variational inclusion, which was introduced and studied by Hassouni and Moudafi [7], is a useful and important extension of the variational inequality. It provides us with a unified, natural, novel, innovative and general technique to study a wide class of problems arising in different branches of mathematical and engineering sciences, see [1-16, 18] and the references therein.

In 2001, Huang and Fang [11] introduced the concept of generalized $m$-accretive mapping, which is a generalization of the $m$-accretive mapping, and studied the properties of the resolvent operator associated with the generalized $m$-accretive mapping in Banach spaces. Furthermore, Huang [10] and Huang-Fang-Deng [13] introduced and studied some new classes of nonlinear variational inclusions involving generalized $m$-accretive mappings in Banach spaces. By using the resolvent operator technique in [11], they constructed some iterative algorithms for solving the nonlinear variational inclusions involving generalized $m$-accrete mappings. They also proved the existence of solution for nonlinear variational inclusions involving generalized $m$-accretive mappings and convergence of sequences generated by the algorithms.

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On the other hand, the sensitivity analysis for variational inequalities and inclusions has been studied by many authors, see, for example, [1, 2, 4, 5, 14-16, 18] and the reference therein.


Inspired and motivated by recent research works in this field, in this paper, we introduce and study a new class of strongly nonlinear quasi-variational inclusions involving generalized $m$-accretive mapping in Banach spaces. By using the resolvent operator technique for generalized $m$-accretive mapping due to Huang and Fang, we prove the existence and uniqueness of solutions, and study the sensitivity analysis for this class of strongly nonlinear quasi-variational inclusions involving generalized $m$-accretive mapping in Banach spaces. Our results improve and generalize the corresponding results of [2, 4, 16].

2. Preliminaries

Let $X$ be a real Banach space with dual space $X^*$; $\langle \cdot, \cdot \rangle$ be the dual pair between $X$ and $X^*$, and $2^X$ denote the family of all the nonempty subsets of $X$. The generalized duality mapping $J_q : X \rightarrow 2^{X^*}$ is defined by

$$J_q(x) = \left\{ f^* \in X^* : \langle x, f^* \rangle = \| f^* \| \| x \| \text{ and } \| f^* \| = \| x \|^{q-1} \}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, $J_2$ is the usual normalized duality mapping. It is known that, $J_q(x) = \| x \|^{q-2}J_2(x)$ for all $x \neq 0$ and $J_q$ is single-valued if $X^*$ is strictly convex [17]. If $X = H$ is a Hilbert space, then $J_2$ becomes the identity mapping of $H$. In the sequel we shall denote the single-valued generalized duality mapping by $j_q$.

**Definition 2.1.** A mapping $N : X \times X \rightarrow X$ is said to be

(i) $\alpha$-strongly accretive with respect to the first argument if there exists a constant $\alpha > 0$ and for any $x, y \in X$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle N(x, \cdot) - N(y, \cdot), j_q(x - y) \rangle \geq \alpha \| x - y \|^q.$$

(ii) $\beta$-Lipschitz continuous with respect to the first argument if there exists a constant $\beta > 0$ such that

$$\| N(x, \cdot) - N(y, \cdot) \| \leq \beta \| x - y \|, \forall x, y \in X.$$

In a similar way, we can define strongly accretive and Lipschitz continuity of $N$ with respect to the second argument.
Definition 2.2. A mapping $\eta : X \times X \to X^*$ is said to be

(i) monotone if
\[ \langle u - v, \eta(u, v) \rangle \geq 0, \quad \forall u, v \in X; \]

(ii) strictly monotone if
\[ \langle u - v, \eta(u, v) \rangle > 0, \quad \forall u, v \in X \]

and equality holds if and only if $u = v$;

(iii) strongly monotone if there exists a constant $\delta > 0$ such that
\[ \langle u - v, \eta(u, v) \rangle \geq \delta \|u - v\|^2, \quad \forall u, v \in X; \]

(iv) Lipschitz continuous if there exists a constant $\tau > 0$ such that
\[ \|\eta(u, v)\| \leq \tau \|u - v\|, \quad \forall u, v \in H. \]

Definition 2.3 [11]. Let $\eta : X \times X \to X^*$ be a single-valued mapping and $M : X \to 2^X$ be a set-valued mapping. Then $M$ is said to be

(i) $\eta$-accretive if
\[ \langle x - y, \eta(u, v) \rangle \geq 0, \quad \forall u, v \in X, x \in Mu, y \in Mv; \]

(ii) strongly $\eta$-accretive if there exists a constant $r > 0$ such that
\[ \langle x - y, \eta(u, v) \rangle \geq r \|u - v\|^2, \quad \forall u, v \in X, x \in Mu, y \in Mv; \]

(iii) generalized $m$-accretive if $M$ is $\eta$-accretive and $(I + \lambda M)(X) = X, \quad \forall \lambda > 0$.

Remark 2.1. Huang and Fang gave one example of generalized $m$-accretive mappings in [11].

Let $\eta : X \times X \to X^*$ be a single-valued mapping and $M : X \to 2^X$ be a set-valued mapping. Then $M$ is said to be

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(ii) strongly $\eta$-accretive if there exists a constant $r > 0$ such that
\[ \langle x - y, \eta(u, v) \rangle \geq r \|u - v\|^2, \quad \forall u, v \in X, x \in Mu, y \in Mv; \]

(iii) generalized $m$-accretive if $M$ is $\eta$-accretive and $(I + \lambda M)(X) = X, \quad \forall \lambda > 0$.

Problem (2.1) is equivalent to the following problem:
\[ 0 \in Au + G(u). \quad (2.2) \]

which is called the nonlinear variational inclusion involving generalized $m$-accretive mapping considered by Bi, Han and Fang [4].
II) If \( X = X^* = H \) is a Hilbert space, \( \eta(u, v) = u - v \), then problem (2.2) becomes the usual variational inclusion with a maximal monotone mapping \( M \).

III) If \( X = X^* = H \) is a Hilbert space, \( \eta(u, v) = u - v, G = \Delta \varphi \), where \( \Delta \varphi \) denotes the subdifferential of a proper convex lower semicontinuous function \( \varphi \) on \( H \), then problem (2.2) becomes the following classical variational inequality:

Find \( u \in H \) such that

\[
\langle Au, v - u \rangle + \varphi(v) - \varphi(u) \geq 0, \forall v \in H.
\] (2.3)

We now establish the parametric version of problem (2.1). To this end, let \( \Omega \) be a nonempty open subset of \( X \) in which the parametric \( \lambda \) takes values. Let \( N : X \times X \times \Omega \to X \) be a nonlinear mapping and \( M : X \times X \times \Omega \to 2^X \) be a generalized \( m \)-accretive mapping with respect to the first argument. The parametric generalized nonlinear variational inclusion problem is to find \( u \in X \) such that

\[
0 \in N(u, u, \lambda) + M(u, u, \lambda).
\] (2.4)

**Remark 2.2.** Since \( M \) is a generalized \( m \)-accretive mapping with respect to the first argument, we define

\[
J_M^{\rho}((\cdot, u, \lambda))(z) = (I + \rho M(\cdot, u, \lambda))^{-1}(z), \quad \forall z \in X,
\]

which is called the resolvent operator associated with \( M(\cdot, u, \lambda) \), where \( \rho > 0 \) is a constant and \( \eta : X \times X \to X^* \) is a strongly monotone mapping.

We also need the following assumption for the resolvent operator \( J_M^{\rho}(\cdot, u, \lambda) \).

**Assumption 2.1.** There is a constant \( \gamma > 0 \) such that

\[
\|J_M^{\rho}((u, v, \lambda))w - J_M^{\rho}((u, v, \lambda))w\| \leq \gamma\|u - v\|,
\]

for all \((u, v, w, \lambda) \in X \times X \times X \times \Omega\).

**Lemma 2.1.** \( u \) is a solution of problem (2.4) if and only if there exists \( u \in X \) such that

\[
u = F(u, \lambda) \overset{\text{def}}{=} J_M^{\rho}((\cdot, u, \lambda))(u - \rho N(u, u, \lambda)),
\]

where \( \rho > 0 \) is a constant.

**Lemma 2.2 [17].** Let \( X \) be a real uniformly smooth Banach space. Then \( X \) is \( q \)-uniformly smooth if and only if there exists a constant \( C_q > 0 \) such that for all \( x, y \in X \),

\[
\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x) \rangle + C_q\|y\|^q.
\]

**Lemma 2.3 [11].** Let \( \eta : X \times X \to X^* \) be strongly monotone and Lipschitz continuous with constants \( \delta > 0 \) and \( \tau > 0 \), respectively. Let \( M : X \to 2^X \) be a generalized \( m \)-accretive mapping. Then the resolvent operator for \( M \) is Lipschitz continuous with constant \( \tau/\delta \), i.e.,

\[
\|J_M^{\rho}(u) - J_M^{\rho}(v)\| \leq \frac{\tau}{\delta}\|u - v\|, \quad \forall u, v \in X.
\]
3. Main results

**Theorem 3.1.** Let $X$ be a $q$-uniformly smooth Banach space and $\eta : X \times X \to X^*$ be strongly monotone and Lipschitz continuous with constants $\delta$ and $\tau$, respectively. Let $N : X \times X \times \Omega \to X$ be a $\alpha$-strongly accretive and $\beta$-Lipschitz continuous single-valued mapping with respect to the first argument, and $M$ is $\xi$-Lipschitz continuous with respect to the second argument. If Assumption 2.1 holds, and there exists $\rho > 0$ such that

$$\theta = \frac{\tau}{\delta}(1 - \rho q \alpha + c_q \rho \beta^3)^{\frac{1}{q}} + \frac{\tau}{\delta} \xi + \gamma < 1,$$

where $C_q > 0$ is the same as in Lemma 2.2. Then the problem (2.4) has a unique solution $u \in X$.

**Proof.** For any given $\lambda \in \Omega$, by Lemma 2.1, we know that $u$ is a solution of the problem (2.4) if and only if $u = J^M_{\rho}(u - \rho N(u, u, \lambda))$, where $\rho$ is a constant.

Letting $F(u, \lambda) = J^M_{\rho}(u - \rho N(u, u, \lambda))$, then $u$ is a solution of the problem (2.4) if and only if $u$ is a fixed point of $F(u, \lambda)$. So we only need to prove that $F(u, \lambda)$ has a unique fixed point in $X$.

In fact, for any $(u, v, \lambda) \in X \times X \times \Omega$, we have

$$F(u, \lambda) = J^M_{\rho}(u - \rho N(u, u, \lambda))$$

and

$$F(v, \lambda) = J^M_{\rho}(v - \rho N(v, v, \lambda)).$$

From the definition of $J^M_{\rho}$, Assumption 2.1 and Lemma 2.3, it follows that

$$||F(u, \lambda) - F(v, \lambda)||$$

$$= ||J^M_{\rho}(u - \rho N(u, u, \lambda)) - J^M_{\rho}(v - \rho N(v, v, \lambda))||$$

$$\leq ||J^M_{\rho}(u - \rho N(u, u, \lambda)) - J^M_{\rho}(v - \rho N(v, v, \lambda))||$$

$$+ ||J^M_{\rho}(v - \rho N(v, v, \lambda)) - J^M_{\rho}(v - \rho N(v, v, \lambda))||$$

$$\leq \frac{\tau}{\delta} \|u - v - \rho(N(u, u, \lambda) - N(v, v, \lambda))\| + \gamma \|u - v\|$$

$$\leq \frac{\tau}{\delta} \|u - v - \rho(N(u, u, \lambda) - N(v, v, \lambda))\|$$

$$+ \rho \|N(v, v, \lambda) - N(v, v, \lambda)\| + \gamma \|u - v\|$$

$$\leq \frac{\tau}{\delta} \|u - v - \rho(N(u, u, \lambda) - N(v, v, \lambda))\|$$

$$+ \frac{\tau}{\delta} \xi \|u - v\| + \gamma \|u - v\|.$$  \hspace{1cm} (3.2)

Since $N$ is $\alpha$-strongly accretive and $\beta$-Lipschitz continuous with respect to the first argument, we have

$$\|u - v - \rho(N(u, u, \lambda) - N(v, v, \lambda))\|^q$$

$$\leq \|u - v\|^q - q \rho \|N(u, u, \lambda) - N(v, v, \lambda), j_q(u - v)\|$$

$$+ C_q \rho \beta^3 \|N(u, u, \lambda) - N(v, v, \lambda)\|^q$$

$$\leq (1 - q \rho \alpha + C_q \rho \beta^3)\|u - v\|^q$$  \hspace{1cm} (3.3)
From (3.2) and (3.3), it follows that
\[ \|F(u, \lambda) - F(v, \lambda)\| \leq \theta \|u - v\|, \quad \forall (u, v, \lambda) \in X \times X \times \Omega, \]
where
\[ \theta = \frac{\tau}{\delta}(1 - q \alpha + C_q \rho^\beta \beta^\gamma) + \frac{\tau}{\delta} \xi + \gamma. \]
From (3.1), we know that \( \theta < 1 \). Thus \( F(u, \lambda) \) is a contractive mapping, and so \( F(u, \lambda) \) has a unique fixed point in \( X \). Therefore problem (2.4) has a unique solution in \( X \). This completes the proof.

**Theorem 3.2.** Let \( X, \eta, N, M \) be the same as in Theorem 3.1. Suppose that Assumption 2.1 and conditions (3.1) of Theorem 3.1 hold. If for any \( u, v, w \in X \), the mapping \( \lambda \to N(u, v, \lambda) \) and \( \lambda \to J^\rho_M(u, \lambda)w \) are both continuous (or Lipschitz continuous) from \( \Omega \) to \( X \), then the solution \( u(\lambda) \) of the parametric problem (2.4) is continuous (or Lipschitz continuous) from \( \Omega \) to \( X \).

**Proof.** For any fixed \( \lambda, \bar{\lambda} \in \Omega \), it follows from Theorem 3.1 and the definition of \( F(u, \lambda) \) that
\[
\|u(\lambda) - u(\bar{\lambda})\| \\
= \|F(u(\lambda), \lambda) - F(u(\bar{\lambda}), \bar{\lambda})\| \\
\leq \|F(u(\lambda), \lambda) - F(u(\bar{\lambda}), \lambda)\| + \|F(u(\bar{\lambda}), \lambda) - F(u(\bar{\lambda}), \bar{\lambda})\| \tag{3.4}
\]
and
\[
\|F(u(\bar{\lambda}), \lambda) - F(u(\bar{\lambda}), \bar{\lambda})\| = \|J^\rho_M(u(\bar{\lambda}), \lambda)(u(\bar{\lambda}) - \rho N(u(\bar{\lambda}), u(\bar{\lambda}), \lambda)) \\
- J^\rho_M(u(\bar{\lambda}), \bar{\lambda})(u(\bar{\lambda}) - \rho N(u(\bar{\lambda}), u(\bar{\lambda}), \bar{\lambda}))\| \\
\leq \|J^\rho_M(u(\bar{\lambda}), \lambda)(u(\bar{\lambda}) - \rho N(u(\bar{\lambda}), u(\bar{\lambda}), \lambda))\| \\
- J^\rho_M(u(\bar{\lambda}), \bar{\lambda})(u(\bar{\lambda}) - \rho N(u(\bar{\lambda}), u(\bar{\lambda}), \bar{\lambda}))\| \\
+ \|J^\rho_M(u(\bar{\lambda}), \lambda)(u(\bar{\lambda}) - \rho N(u(\bar{\lambda}), u(\bar{\lambda}), \bar{\lambda}))\| \\
- J^\rho_M(u(\bar{\lambda}), \bar{\lambda})(u(\bar{\lambda}) - \rho N(u(\bar{\lambda}), u(\bar{\lambda}), \bar{\lambda}))\| \tag{3.5}
\]
It follows from (3.4) and (3.5) that
\[
\|u(\lambda) - u(\bar{\lambda})\| \leq \frac{\rho \tau}{(1 - \theta)\delta} \|N(u(\bar{\lambda}), u(\bar{\lambda}), \lambda) - N(u(\bar{\lambda}), u(\bar{\lambda}), \bar{\lambda})\| \\
+ \frac{1}{\tau} \|J^\rho_M(u(\bar{\lambda}), \lambda)(u(\bar{\lambda}) - \rho N(u(\bar{\lambda}), u(\bar{\lambda}), \lambda)) \\
- J^\rho_M(u(\bar{\lambda}), \bar{\lambda})(u(\bar{\lambda}) - \rho N(u(\bar{\lambda}), u(\bar{\lambda}), \bar{\lambda}))\|.
\]
This completes the proof.
Remark 3.1. Theorems 3.1 and 3.2 extend and improve the corresponding results in [2, 4, 16].

References