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SUBDIFFERENTIABILITY OF TYPICAL CONTINUOUS FUNCTIONS

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Abstract. We prove a "typical" subdifferentiability principle and apply it to a variety of complete metric spaces of continuous functions on separable Banach spaces; so as to obtain existence of functions with maximal subdifferentials when ordered by inclusion. The relationship between continuous functions with maximal subdifferentials and nowhere monotone functions is discussed.

1. Introduction

The Clarke subdifferential and the approximate subdifferential have proved to be flexible tools in analysis and in optimization. It is our goal in this note to show that most continuous functions have maximal subdifferentials. In §3 using the Baire Category Theorem we prove *a typical subdifferentiability principle*, which turns out to be a surprisingly effective tool for studying generalized subderivatives in a variety of complete metric spaces. In §4 we deduce the existence of *maximal* subdifferentials in appropriate complete metric spaces. In §5 we show how *nowhere monotone functions of the second species* can be used to obtain continuous functions with maximal subdifferentials.

2. Zoo of subdifferentials

Throughout, we assume that X is a separable Banach space whose dual with weak* topology is denoted by (X^*, w^*), $B_\delta[x] := \{y \in X : \|x - y\| \leq \delta\}$, $B_X := \{x \in X : ||x|| \leq 1\}$; $B_{X^*} := \{x^* \in X^* : ||x^*|| \leq 1\}$. For $C \subset X^*$, its w*-closure is denoted by $w^*\mathrm{cl}C$, its w*-closed convex hull by $\overline{co}^{w^*} C$, its norm interior by $\mathrm{int}(C)$, and we define the *support function* of C as $\sigma_C(x) := \sup_{c\in C}\langle x, c\rangle$ for $x \in X$. Let $f : X \to (-\infty, +\infty]$ be lower semicontinuous and $f(x)$ finite. By $u \to_f x$ we mean $u \to x, f(u) \to f(x)$. For $S \subset X$, we set

$$I_S(x) := \begin{cases} 0 & \text{if } x \in S, \\ +\infty & \text{if } x \notin S. \end{cases} \qquad f_S(x) := \begin{cases} f(x) & \text{if } x \in S, \\ +\infty & \text{if } x \notin S. \end{cases}$$

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We will use the following subdifferentials:

The *Dini-Hadamard* subdifferential of f at x is defined by:

$$\partial_- f(x) := \{x^* \in X^* : \langle x^*, v\rangle \leq f^-(x;v) \text{ for all } v \in X\},$$

where $f^-(x;v) := \liminf_{\substack{t\downarrow 0\\ h\to v}} \frac{f(x+th)-f(x)}{t}$.

The *Ioffe-approximate subdifferential* of f at x is defined by:

$$\partial_a f(x) := \bigcap_{L\in\mathcal{F}} \limsup_{u\to_f x} \partial^- f_{u+L}(u),$$

where $\mathcal{F}$ denotes the collection of all finite dimensional subspaces of X and lim sup denotes the collection of limits of converging subnets. It follows from the definition that $\partial_a f(x)$ is w*-closed and $\partial_a f(x) = \limsup_{u\to_f x} \partial_a f(u)$. When X is finite dimensional, it reduces to the *Mordukhovich-Ioffe approximate subdifferential* defined by:

$$\partial_a f(x) := \bigcap_{\delta>0} \operatorname{cl}\{z^* \in X^* : z^* \in \partial_- f(z), \|z-x\| < \delta, |f(z)-f(x)| < \delta\}.$$

Suppose $x \in \operatorname{cl} S \subset X$. Let d_S denote the distance function associated with S. The *G-normal cone* to S at x is:

$$N_G(S,x) := \text{w*cl}\bigcup\{\lambda\partial_a d_S(x) : \lambda > 0\}.$$

The *G-subdifferential* and *singular G-subdifferential* of f at x are defined by:

$$\partial_g f(x) := \{x^* \in X^* : (x^*,-1) \in N_G(epi\ f, (x, f(x)))\},$$
$$\partial_g^\infty f(x) := \{x^* \in X^* : (x^*,0) \in N_G(epi\ f, (x, f(x)))\},$$

where $epi\ f := \{(y,\beta) \in X\times\mathbb{R} : \beta \geq f(y)\}$. When X is finite dimensional, $\partial_g f$ and $\partial_a f$ are the same. If f is directionally Lipschitz at x, then $\partial_g f(x) = \partial_a f(x)$. In general, $\partial_g f(x) \subset \partial_a f(x)$.

The *Gâteaux-viscosity subdifferential of rank* k of f at x is the set $D_g^k f(x)$ of vector $x^* \in X^*$ with the property that there is a function m (depending on x^*) which is Gâteaux-smooth on a neighborhood U of x and satisfies: (i) m satisfies a Lipschitz condition with constant k on U; (ii) $m(x) = f(x)$; (iii) $\nabla m(x) = x^*$, (iv) $f(u) \geq m(u)$ for all $u \in U$. On separable Banach space, by [1] one has:

$$\partial_g f(x) = w^*\text{cl}\bigcup_{k=1}^{\infty}\{w^* - \lim_{n\to\infty} x_n^* : x_n^* \in D_g^k f(x_n), x_n \to_f x\}. \tag{2.1}$$

The *Clarke subdifferential* of f at x is defined by:

$$\partial_c f(x) := \{x^* \in X^* : \langle x^*, v\rangle \leq f^\uparrow(x,v) \text{ for all } v \in X\}, \quad \text{where}$$
$$f^\uparrow(x;v) := \lim_{\epsilon\downarrow 0} \limsup_{\substack{y\to_f x\\ t\downarrow 0}} \inf_{w\in v+\epsilon B_X} \frac{f(y+tw)-f(y)}{t}.$$

The set $\partial_c f(x)$ is w*-closed and convex (possibly empty). If $f : X \to (-\infty, +\infty]$ is lower semicontinuous, then f is *Clarke subdifferentiable* at a dense subset of points in its graph, but the multifunction $\partial_c f$ need not be closed [13], in the sense that: $x_k \to x$, $f(x_k) \to f(x)$, $z^k \in \partial_c f(x_k), z^k \to z \Rightarrow z \in \partial_c f(x)$. In general, $\partial_c f(x) = \overline{co}^{w^*}(\partial_g f(x) + \partial_g^\infty f(x))$ [10].

3. A typical subdifferentiability principle

Theorem 3.1. *Let A be an open subset of a separable Banach space X, and (Y, ρ) a complete metric space of real-valued continuous functions on A such that the metric ρ is given by*

(i) $\rho(g_1, g_2) := \inf\{1, \sup_{x\in A} |g_1(x) - g_2(x)|\}$ *for all* $g_1, g_2 \in Y$*;*

(ii) *For all* $g_1, g_2 \in Y$, $r > 0$, $\min\{g_1, g_2\} \in Y$, $\max\{g_1 - r, g_2\} \in Y$*;*

(iii) *There exist a w^*-compact convex subset $C \subset X^*$ having nonempty norm interior such that for each $x \in A$ and $r \in \mathbb{R}$ there exists a $g_x \in Y$ with $g_x(\tilde{x}) = \sigma_C(\tilde{x} - x) + r$ for $\tilde{x}$ near by x.*

Then the set $\{f \in Y : \partial_a f(x) \supset \partial_g f(x) \supset C$ for all $x \in A\}$ is residual in (Y, ρ).

Proof. Fix $x \in A$, $c \in \mathrm{int}(C)$, $n > 1$. Consider

$$G^n_{x,c} := \Big\{ f \in Y : \text{there exists} \tilde{x} \in A \ \text{ satisfying } \ \|\tilde{x} - x\| < \frac{1}{n} \ \text{ such that} \\ \text{for some } \ \tilde{x}^* \in \partial_- f(\tilde{x}) \ \text{ we have } \ \|\tilde{x}^* - c\| < \frac{1}{n} \Big\}.$$

(a) *int$(G^n_{x,c})$ is dense in Y.* Choose an arbitrary $f \in Y$ and $1 > 3\epsilon > 0$. By (iii) there exists a $h \in Y$ such that $h(\tilde{x}) := f(x) - \epsilon + \sigma_C(\tilde{x} - x)$ for $\tilde{x}$ near by x. Define $h_1 := \min\{f, h\}$ and $h_2 := \max\{f - 2\epsilon, h_1\}$. Then $\rho(f, h_2) < 3\epsilon$, and $h_2 \in Y$ by (ii). For $0 < \delta < 1/n$ sufficiently small, we have

$$h_2(\tilde{x}) = h(\tilde{x}) = f(x) - \epsilon + \sigma_C(\tilde{x} - x) \ \text{ on } \ B_\delta[x].$$

As h is convex on $B_\delta[x]$, $\partial_- h_2(x) = C$, we see that $h_2 \in G^n_{x,c}$. We will show that $h_2 \in \mathrm{int}(G^n_{x,c})$. Because $c \in \mathrm{int}(C)$, we may assume $rB_{X^*} \subset C - c$ for some $r > 0$. On $B_\delta[x]$, we have

$$\begin{aligned} m &:= \inf\{h_2(\tilde{x}) - \langle c, \tilde{x}\rangle : \|\tilde{x} - x\| = \delta\} \\ &= h_2(x) - \langle c, x\rangle + \inf\{\sigma_{C-c}(\tilde{x} - x) : \|\tilde{x} - x\| = \delta\} \\ &\geq h_2(x) - \langle c, x\rangle + r\delta > h_2(x) - \langle c, x\rangle. \end{aligned}$$

Let $\alpha := m - (h_2(x) - \langle c, x\rangle) > 0$. For $g \in Y$ with $\rho(g, h_2) < \beta < \min\{1, \alpha/2\}$, for $\|\tilde{x} - x\| = \delta$ we have $g(\tilde{x}) - \langle c, \tilde{x}\rangle = g(\tilde{x}) - h_2(\tilde{x}) + h_2(\tilde{x}) - \langle c, \tilde{x}\rangle \geq -\beta + m$, and

$$g(x) - \langle c, x\rangle = g(x) - h_2(x) + h_2(x) - \langle c, x\rangle \leq \beta + h_2(x) - \langle c, x\rangle.$$

Then

$$\begin{aligned} \inf\{g(\tilde{x}) - \langle c, \tilde{x}\rangle : \|\tilde{x} - x\| = \delta\} &\geq -\beta + m \\ &> \beta + h_2(x) - \langle c, x\rangle \geq g(x) - \langle c, x\rangle. \end{aligned} \tag{3.1}$$

Define $g_1 := g - \langle c, \cdot\rangle + I_{B_\delta[x]}$, which is lower semicontinuous and bounded below on X. By Equation (3.1), we have

$$\begin{aligned} \inf_X g_1 &= \inf_{B_\delta[x]} (g(\tilde{x}) - \langle c, \tilde{x}\rangle) \\ &\leq g(x) - \langle c, x\rangle < \inf\{g(\tilde{x}) - \langle \tilde{x}, c\rangle : \|\tilde{x} - x\| = \delta\}. \end{aligned}$$

Let $0 < \nu < \min\{1/(2n), \inf\{g(\tilde{x}) - \langle \tilde{x}, c\rangle : \|\tilde{x} - x\| = \delta\} - (g(x) - \langle x, c\rangle)\}$. Choose x_0 such that

$$g_1(x_0) = g(x_0) - \langle c, x_0\rangle < \inf_X g_1 + \nu.$$

Since X is separable, X is smoothable. By the Borwein-Preiss smooth variational principle [3] there exists a Gâteaux differentiable ϕ with Lipschitz constant on X as small as one wishes and $v \in X$ such that for all $\tilde{x} \in B_\delta[x]$,

$$g(\tilde{x}) - \langle c, \tilde{x}\rangle + \phi(\tilde{x}) \geq g(v) - \langle c, v\rangle + \phi(v), \tag{3.2}$$

$$g(v) - \langle c, v\rangle < \nu + \inf_X g_1, \tag{3.3}$$

$$\|\nabla\phi(v)\| \leq \sup\{\|\nabla\phi(x)\| : x \in X\} < 2\nu < \frac{1}{n}. \tag{3.4}$$

Equation (3.3) shows $g(v) - \langle c, v\rangle < \inf\{g(\tilde{x}) - \langle c, \tilde{x}\rangle : \|\tilde{x} - x\| = \delta\}$, thus $\|v - x\| < \delta < 1/n$. Equation (3.2) shows $0 \in \partial_- g(v) - c + \nabla\phi(v)$, thus there exist $v^* \in \partial_- g(v)$ with $\|v^* - c\| = \|\nabla\phi(v)\| < 1/n$ by Equation (3.4). Then $g \in G^n_{x,c}$, and so $B_\beta(h_2) \subset G^n_{x,c}$, as required. [When X is finite dimensional, we may choose $\phi = 0$ by the compactness of $B_\delta[x]$.]

(b) Since $\operatorname{int}(G^n_{x,c})$ is open and dense in Y, the set $G_{x,c} := \bigcap_{n=1}^\infty \operatorname{int}(G^n_{x,c})$ is dense in Y. If $f \in G_{x,c}$, then for every n there exists $x_n^* \in \partial_- f(x_n)$ with $\|x_n^* - c\| < 1/n$ and $\|x_n - x\| < 1/n$. Letting $n \to \infty$, we obtain $c \in \partial_a f(x)$.

(c) Since C is w*-compact convex with non-empty norm interior, we may take a countable dense set $\{c_k\}$ from $\operatorname{int}(C)$ such that $C = w^*\mathrm{cl}\{c_k : k \in \mathbb{N}\}$. As G_{x,c_k} is dense G_δ for each c_k, we have $G_x := \bigcap_{k=1}^\infty G_{x,c_k}$ is a dense G_δ set in Y. If $f \in G_x$, then $c_k \in \partial_a f(x)$ for every k by (b), thus $C \subset \partial_a f(x)$ as $\partial_a f(x)$ is w*-closed.

(d) Now let $\{x_k\}$ be a countable norm dense set in A. Since G_{x_k} is dense G_δ for every k, the set $G := \bigcap_{k=1}^\infty G_{x_k}$ is a dense G_δ set in Y. If $f \in G$, for every k we have $\partial_a f(x_k) \supset C$ by (c). By w*-upper semicontinuity of $\partial_a f$, we get $\partial_a f(x) \supset \limsup_{x_k \to x} \partial_a f(x_k) \supset C$. Hence $\partial_a f \supset C$ for every $f \in G$.

(e) Finally we compute $\partial_g f$ for $f \in G$. Equations (3.2) and (3.4) in fact show that $c - \nabla\phi(v) \in D^k_g f(v)$ with $k := \sup\{\|c\| + 1 : c \in C\}$. From (b) we see that if $f \in G_{x,c}$ then for every n there exists $x_n^* \in D^k_g f(x_n)$ with $\|x_n^* - c\| < 1/n$ and $\|x_n - x\| < 1/n$. In other words, for every neighbourhood U of x, for every $\epsilon > 0$ there exists $\|x_\epsilon - x\| < \epsilon$ and $x_\epsilon^* \in D^k_g f(x_\epsilon)$ such that $\|x_\epsilon^* - c\| < \epsilon$. From (c) we see that if $f \in G_x$ then for every c_m this holds. From (d) we see that this holds for all $\{c_m : m \in \mathbb{N}\}$ at every x_k if $f \in G$.

Let $f \in G$. Fix c_m and x. For every $B_{1/n}(x)$ there exists $\|x_k - x\| < 1/n$ as $\{x_k\}_{k=1}^\infty$ is dense in A. As $f \in G_{x_k,c_m}$, there exists $\|v_n - x_k\| < 1/n - \|x_k - x\|$ and $v_n^* \in D^k_g f(v_n)$ such that $\|v_n^* - c_m\| < 1/n$. That is, there exists v_n and v_n^* such that $v_n^* \in D^k_g f(v_n)$ and $\|v_n^* - c_m\| < 1/n$. Letting $n \to \infty$, we obtain $c_m \in \partial_g f(x)$ by (2.1). As m is arbitrary, $\bigcup_{m=1}^\infty \{c_m\} \subset \partial_g f(x)$. Because $\{c_m\}_{m=1}^\infty$ is w*-dense in C and $\partial_g f(x)$ is w*-closed, we obtain $C \subset \partial_g f(x)$.

When Y is a subset of bounded continuous functions defined on A, we may replace ρ by supremum metric defined by $\rho_\infty(g_1, g_2) := \sup_{x\in A} |g_1(x) - g_2(x)|$ for $g_1, g_2 \in Y$.

4. The existence of maximal subdifferentials

We now apply Theorem 3.1 to various choices of complete metric spaces Y. Let $C(A)$ denote the space of all *continuous functions* on A with metric ρ.

Corollary 4.1. *If A is an open subset of a separable Banach space X, then in $C(A)$ the set $\{f \in C(A) : \partial_c f = \partial_g f = \partial_a f = X^*$ on $A\}$ is residual.*

Proof. or each n the set $G_n := \{f \in C(A) : \partial_a f \supset \partial_g f \supset nB_{X^*}\}$, is residual in $C(A)$. The set $G := \bigcap_{n=1}^{\infty} G_n$ is a residual set in $C(A)$. If $f \in G$, for every n we have $\partial_a f(x) \supset \partial_g f(x) \supset nB_{X^*}$ for $x \in A$, then $\partial_a f(x) \supset \partial_g f(x) \supset \bigcup_{n=1}^{\infty} nB_{X^*} = X^*$.

Let $K \subset X$ be a closed convex cone. We say that f is *nondecreasing with respect to K* if $f(x^{'}) \leq f(x^{''})$ when $x^{''} - x^{'} \in K$. We use $C_K(A)$ to denote continuous functions on A which are K-nondecreasing. $(C_K(A), \rho)$ is a complete metric space and is closed under lattice operations. Define the *polar cone* to K by

$$K^+ := \{x^* \in X^* : \langle x^*, x\rangle \geq 0 \text{ for all } x \in K\}.$$

If $f \in C_K(A)$, from the definitions we see that $\partial_c f(x) \subset K^+$ and $\partial_a f(x) \subset K^+$.

Corollary 4.2. *If K^+ has nonempty norm interior in the dual of a separable Banach space X, then the set $G := \{f \in C_K(A) : \partial_g f = \partial_a f = \partial_c f \equiv K^+$ on $A\}$ is residual in $C_K(A)$. If, in addition, K has non-empty interior, then $\partial_c(-f) \equiv -K^+$ for every $f \in G$.*

Proof. Since K^+ has nonempty norm interior, $K^+ \cap nB_{X^*}$ has nonempty norm interior. For each $n \in \mathbb{N}$, $\sigma_{K^+ \cap nB_{X^*}}(\cdot - x) + r \in C_K(A)$ for each $x \in A, r \in \mathbb{R}$. By Theorem 3.1 each

$$G_n := \{f \in C_K(A) : (K^+ \cap nB_{X^*}) \subset \partial_g f \subset \partial_a f\}$$

is residual in $C_K(A)$. If $f \in G := \bigcap_{n=1}^{\infty} G_n$, then $K^+ \subset \partial_g f \subset \partial_a f$, and so $\partial_c f = \partial_a f = \partial_g f \equiv K^+$. When K has non-empty norm interior, each $f \in C_K(A)$ is directionally Lipschitz, thus $\partial_c(-f) = -\partial_c f$ [12].

By [14], each lower semicontinuous and nowhere locally Lipschitz function f on A is Gâteaux-viscosity subdifferentiable at most on a first category subset of A. If $f \in C_K(A)$ satisfies $\partial_c f(x) = K^+$ for each $x \in A$, then f and $-f$ are nowhere locally Lipschitz on A, thus both f and $-f$ are Gâteaux-viscosity subdifferentiable at most on a first category subset of A. When X is finite dimensional, each such f is nowhere directionally Lipschitz on A by [12]. Also, when $K = 0$, Corollary 4.2 recovers Corollary 4.1.

Corollary 4.3. *Let S be a closed subset of a separable Banach space X and $f : S \to \mathbb{R}$ be continuous. Then there exist many continuous function $g : X \to \mathbb{R}$ such that $g|_S = f$ and $\partial_c g = \partial_a g = \partial_g g = X^*$ on $X \setminus S$.*

Proof. Let $A := X \setminus S$. For each $x \in A$, $r \in \mathbb{R}$ and $n \in \mathbb{N}$, for some $\delta > 0$ we have $B_\delta[x] \subset A$, on this closed ball we define $n\|\tilde{x} - x\| + r$. By Tietze's extension theorem, there exists a g continuous on X such that $g|_S = f$ and $g|_{B_\delta[x]} = n\|\cdot - x\| + r$. Endowed with metric ρ, the space $\{g : g|_S = f, \text{ and } g \text{ is continuous on } X\}$ is complete and satisfies the assumptions of Theorem 3.1.

Corollary 4.4. *Let A be an open subset of a separable Banach space X and $C \subset X^*$ be a w*-compact convex subset with non-empty norm interior. Define*

$$\mathcal{X}_C := \{f : A \to \mathbb{R} : f \text{ is locally Lipschitz on } A \text{ and } \partial_c f(x) \subset C \text{ for every } x \in A\}.$$

Then the set $G := \{f \in \mathcal{X}_C : \partial_a f = \partial_g f = \partial_c f \equiv C \text{ on } A\}$ is a residual set in $(\mathcal{X}_C, \rho)$.

Proof. By Lebourg's mean value theorem [5], one may verify that $(\mathcal{X}_C, \rho)$ is complete. By Propositions 2.3.12, 2.3.3 [5], we have $\partial_c \max\{f, g\} \subset C$, $\partial_c \min\{f, g\} \subset C$ for $f, g \in \mathcal{X}_C$. Hence $(\mathcal{X}_C, \rho)$ satisfies the assumptions of Theorem 3.1.

When X^* is separable and f is locally Lipschitz on A, the *Mordukhovich-Shao sequential subdifferential* $\partial_{ms} f$ coincides with $\partial_a f$ [11]. Hence Corollary 4.4 also holds for $\partial_{ms} f$ when X^* is separable. Since each locally Lipschitz is pseudo-regular generically [9], each $f \in G$ is Gâteaux differentiable at most on a first category subset of A.

5. Nowhere monotone functions on the line

On the line, functions with maximal subdifferentials are closely related to *nowhere monotone functions* as introduced by Garg [8]. Our arguments thence may be made more transparent.

We recall that a finite real function f defined on $[0, 1]$ is *nowhere monotone* if f is not monotone in any subinterval of $[0, 1]$. A nowhere monotone function f is *of the second species* in $[0, 1]$ provided that for every $r \in \mathbb{R}$ the function $f(x) + r \cdot x$ is also nowhere monotone. Every nondifferentiable function f is a nowhere monotone function of the second species. If $f : [0, 1] \to \mathbb{R}$ is continuous and nowhere monotone, there exists a residual set $G \subset [0, 1]$ such that $f_-(x) \le 0 \le f^-(x)$ and $f_+(x) \le 0 \le f^+(x)$ if $x \in G$ [7], and the set of points at which f attains local minima is dense in $(0, 1)$. Let $r \in \mathbb{R}$ and define f_r by $f_r(x) := f(x) + rx$. If f is a nowhere monotone continuous function of the second species on $[0, 1]$, for each $n \in \mathbb{N}$, the set

$$G_n := \{x : f_-(x) \le -n < n \le f^-(x) \text{ and } f_+(x) \le -n < n \le f^+(x)\},$$

is residual. The set $G := \bigcap_{n=1}^\infty G_n$ is residual in $[0, 1]$, and at $x \in G$ we have

$$f^+(x) = f^-(x) = +\infty \text{ and } f_+(x) = f_-(x) = -\infty,$$

thus f is Dini subdifferentiable at most on a first category subset of $[0, 1]$. In the following, by 'prevalent' we mean the complement of a Haar null set (as discussed in [2]).

Theorem 5.1. *The set*

$$D := \{f \in C[0,1] : \partial_a f = \partial_c f \equiv \mathbb{R} \text{ and } \partial_- f \text{ exists only on a first category set}\}.$$

is prevalent and residual in $(C[0,1], \rho_\infty)$.

Proof. If f is not nowhere monotone of the second species on $[0,1]$, then for some r we have f_r monotone on some subinterval $I \subset [0,1]$. Let I be a subinterval of $[0,1]$. Define

$$A_I := \{f \in C[0,1] : \text{ there exists a } r \in \mathbb{R} \text{ with } f_r \text{ being nondecreasing on } I\}.$$

For each $n \in \mathbb{N}$, let A_n denote those functions $f \in C[0,1]$ for which there exists $r \in [-n,n]$ such that f_r is nondecreasing on I. Then $A_I = \bigcup_{n=1}^{\infty} A_n$. We show that for each $n \in \mathbb{N}$ the set A_n is closed and $C[0,1] \setminus A_n$ is dense.

To verify that A_n is closed, let f_k be a sequence of functions in A_n such that $f_k \to f$ uniformly. Then $f \in C[0,1]$, and we must show that $f \in A_n$. For each k, there exists $r_k \in [-n,n]$ such that $f_k(x) + r_k x \geq f_k(y) + r_k y$ if $x \geq y$ and $x, y \in I$. There exists an increasing sequence k_i from $\mathbb{N}$ such that $\{r_{k_i}\}$ converges to some $r \in [-n,n]$. Then $f(x) + rx \geq f(y) + ry$. Thus $f \in A_n$, and A_n is closed in $C[0,1]$. To show that A_n is nowhere dense, we verify that A_n has no interior. Take a continuous nowhere differentiable function g defined on $[0,1]$. For every $\epsilon > 0$, we claim $f + \epsilon g \notin A_n$ if $f \in A_n$. Suppose $f + \epsilon g \in A_n$, then for some r_1 we have $h(x) := f(x) + \epsilon g + r_1 x$ being monotone on I. Since $f \in A_n$, there exists another r_2 with $f(x) + r_2 x$ being monotone on I. But

$$h(x) - r_1 x + r_2 x = (f(x) + r_2 x) + \epsilon g(x),$$

implies $g(x) = [h(x) - (f(x) + r_2 x) - r_1 x + r_2 x]/\epsilon$. Then g is differentiable almost everywhere on I, a contradiction. Thus A_n is nowhere dense and closed.

Now we show that A_n is Haar null. Let g be a nowhere differentiable function. Define a probability measure on Borel subsets of $C[0,1]$ by: $\mu(E) = \lambda\{t \in [0,1] : tg \in E\}$, where λ denotes the Lebesgue measure on $\mathbb{R}$. We will verify $\mu(f + A_n) = 0$ for every $f \in C[0,1]$. In fact, the set $\{t \in [0,1] : tg \in A_n + f\}$ is either empty or singleton. If not, we may find $t_1 \neq t_2$ such that $t_1 g \in f + A_n$ and $t_2 g \in f + A_n$. Then there exists $r_1, r_2 \in [-n,n]$ such that $h_1(x) := t_1 g(x) - f(x) + r_1 x$ and $h_2(x) := t_2 g(x) - f(x) + r_2 x$ are nondecreasing on I. It follows that $g(x) = [h_1(x) - h_2(x) - (r_1 - r_2)x]/(t_1 - t_2)$ is differentiable almost everywhere on I, a contradiction.

Since $A_I = \bigcup_{n=1}^{\infty} A_n$, A_I is Haar null and a countable union of nowhere dense closed sets. The same is true of the set $B_I := \{f \in C[0,1] : -f \in A_I\}$.

Let $\{I_k\}$ be all the subintervals of $[0,1]$ with rational endpoints. Define $A := \bigcup_k A_{I_k}$ and $B := \bigcup_k B_{I_k}$. It follows that each of A and B is Haar null and a countable union of nowhere dense closed subsets in $C[0,1]$. Then $C[0,1] \setminus (A \cup B)$ is a residual set of type G_δ and $A \cup B$ is Haar null. If $f \in C[0,1] \setminus (A \cup B)$, then for every $r \in \mathbb{R}$ the function f_r is not monotonic at any subinterval of $[0,1]$, thus nowhere monotonic of the second species. Each nowhere monotonic function of the second species f has $\partial_a f = \partial_c f \equiv \mathbb{R}$, and $\partial_- f$ exists only on a first category set of $[0,1]$.

Example 1. Nondifferentiable functions constitute a proper subclass of the class of continuous nowhere monotone functions of the second species.

To see this, let us write the interval $[0,1]$ as a disjoint union of measurable sets, $[0,1] = \bigcup_{k=1}^{\infty} B_k$, each with positive measure in every nondegenerate subinterval of $[0,1]$ [4].

If $A := \{a_1, a_2, \dots\}$ is any sequence of real numbers, then there exists an absolutely continuous function F such that for every interval $I \subset [0,1]$ and every k, the set $\{x : F^{'}(x) = a_k\} \cap I$ has positive measure. Indeed, we may assume that $|a_k|\mu(B_k) < 1/k^2$ for each $k > 1$. It follows that the function f defined by $f(x) := a_k$ if $x \in B_k$ is Lebesgue integrable, since

$$\int_0^1 |f(x)|dx \leq |a_1|\mu(B_1) + \sum_{k=2}^{\infty} \frac{1}{k^2} < +\infty.$$

Let F be defined by $F(x) := \int_0^x f(t)dt$. Then F is absolutely continuous and $F^{'}(x) = f(x)$ almost everywhere in $[0,1]$. In particular for each k, $F^{'}$ takes on the value a_k at almost all points of B_k. When $A := \{r \in \mathbb{R} : r \text{ is rational}\}$, such F is absolutely continuous and satisfies $\partial_a F = \partial_c F \equiv \mathbb{R}$ on $[0,1]$.

Consider $Y := \{f : f \text{ is continuous and nondecreasing on } [0,1]\}$ with supremum metric ρ_∞.

Theorem 5.2. *In (Y, ρ_∞), the set $\{f \in Y : \partial_c f = \partial_a f \equiv [0, +\infty)\}$ is a residual set.*

Proof. Let I denote an open subinterval of $[0,1]$, and let

$$A_I^n := \{f \in Y : \text{ there exists } \nu \in [1/n, n] \text{ such that } f_{-\nu} \text{ is nondecreasing on } I\},$$
$$B_I^n := \{f \in Y : \text{ there exists } \nu \in [1/n, n] \text{ such that } f_{-\nu} \text{ is nonincreasing on } I\}.$$

(1) A_I^n is closed. Assume $\{f_m\} \subset A_I^n$ is Cauchy. Then $f_n \to f$ uniformly for some $f \in Y$. For each k, there exists $\nu_k \in [1/n, n]$ such that $f_k(x) - \nu_k x \geq f_k(y) - \nu_k y$ for all $x \geq y$ with $x, y \in I$. There exists an increasing sequence $\{k_i\}$ such that $\{\nu_{k_i}\}$ converges to some $\nu \in [1/n, n]$. Taking limits, we have $f(x) - \nu x \geq f(y) - \nu y$ for $x \geq y$ with $x, y \in I$. Similar arguments show that B_I^n is closed.

(2) To show that A_I^n is nowhere dense, with $f \in Y$ we verify that every open ball $B_{3\epsilon}(f)$ contains points of $Y \setminus A_I^n$. Fix $x_0 \in I$, and define a nondecreasing h by $h(x) := f(x_0) + \epsilon + \min\{x - x_0, 0\}$. Then $h_1 := \max\{f, h\}$ and $h_2 := \min\{f + 2\epsilon, h_1\}$ are continuous and nondecreasing. As $h_1 \geq f$, $f + 2\epsilon \geq f$, we have $f + 2\epsilon \geq h_2 \geq f$. For $\delta > 0$ sufficiently small, we have $h_2(y) = f(x_0) + \epsilon$ for $x_0 \leq y \leq x_0 + \delta$. For every $\nu \in [1/n, n]$, on $[x_0, x_0 + \delta]$ we have $(h_2(y) - \nu \cdot y)^{'} = -\nu < 0$ almost everywhere, thus $h_2(y) - \nu y$ is decreasing on $[x_0, x_0 + \delta]$, and $h_2 \notin A_I^n$.

To show that B_I^n is nowhere dense, we use similar arguments. Define $h \in Y$ by $h(x) := \max\{(n+1)(x - x_0), 0\} + f(x_0) - \epsilon$, $h_1 := \min\{f, h\}$, and $h_2 := \max\{f - 2\epsilon, h_1\}$. Then $h_2 \in Y$ and $f - 2\epsilon \leq h_2 \leq f$. For $\delta > 0$ sufficiently small, $h_2(x) = (n+1)(x - x_0)$ on $[x_0, x_0 + \delta]$. For every $\nu \in [1/n, n]$, $(h_2(x) - \nu \cdot x)^{'} = n + 1 - \nu > 0$ almost everywhere, thus $h_2(x) - \nu \cdot x$ is increasing on $[x_0, x_0 + \delta]$, and $h_2 \notin B_I^n$.

(3) Thus both A_I^n and B_I^n are nowhere dense and closed. The sets $A_I := \bigcup_{n=1}^{\infty} A_I^n$ and $B_I := \bigcup_{n=1}^{\infty} B_I^n$ are first category of type F_σ in Y. Let $\{I_k\}$ be all open subintervals of $[0,1]$ having rational endpoints. The sets $A := \bigcup_{k=1}^{\infty} A_{I_k}$ and $B := \bigcup_{k=1}^{\infty} B_{I_k}$ are first category of type F_σ. It follows that the set $Y \setminus (A \cup B)$ is a residual set of type G_δ. If $f \in Y \setminus (A \cup B)$, then for every $\nu > 0$, the function $f_{-\nu}$ is not monotonic on every I_k, thus nowhere monotonic on $[0,1]$. The set of points at which $f_{-\nu}$ attains local minimum is dense in $[0,1]$. We have $\nu \in \partial_a f(x)$ for every $x \in [0,1]$. Since $\nu \in (0,+\infty)$ is arbitrary, we have $[0,+\infty) \subset \partial_a f(x) \subset \partial_c f(x) \subset [0,+\infty)$.

Finally we consider $Lip_1 := \{f : [0,1] \to \mathbb{R} : |f(x) - f(y)| \le |x - y| \text{ for all } x, y \in [0,1]\}$ with supremum metric ρ_∞.

Theorem 5.3. *In* (Lip_1, ρ_∞), *the set*

$$G := \{f : f(x) - r \cdot x \text{ is nowhere monotone on } [0,1] \text{ for every } |r| < 1\}$$

is residual.
In particular, for $f \in G$ *we have* $\partial_c f = \partial_a f \equiv [-1,1]$.

Proof. Let I denote an open subinterval of $[0,1]$, and let

$$A_I^n := \{f \in Lip_1 : \text{there exists some } r \in [-1 + 1/n, 1 - 1/n] \text{ with } f(x) - r \cdot x \text{ being nondecreasing on } I\}.$$

To verify that A_I^n is closed, let $\{f_k\}$ be a sequence of functions in A_I^n such that $f_k \to f$ uniformly. Then $f \in Lip_1$, and we must show that $f \in A_I^n$. For each $k \in \mathbb{N}$, there exists $r_k \in [-1 + 1/n, 1 - 1/n]$ such that $f_k(x) - r_k x \ge f_k(y) - r_k y$ for $x \ge y \in I$. There exists an increasing sequence $\{k_i\}$ from $\mathbb{N}$ such that $\{r_{k_i}\}$ converges to some $r \in [-1 + 1/n, 1 - 1/n]$, then $f(x) - rx \ge f(y) - ry$ for $x \ge y \in I$. Then $f \in A_I^n$ and A_I^n is closed in Lip_1.

To show that A_I^n is nowhere dense, we verify that every ball in Lip_1 contains points of $Lip_1 \setminus A_I^n$. Let $B_{5\epsilon}(f)$ be an open ball in Lip_1. If $f \notin A_I^n$, there is nothing to prove, so assume $f \in A_I^n$. Let $(x_0 - \epsilon, x_0 + \epsilon) \subset I$. We define

$$\phi_\epsilon(t) := \begin{cases} -1 & \text{if } t \in (x_0 - \epsilon, x_0], \\ 1 & \text{if } t \in (x_0, x_0 + \epsilon], \\ f'(t) & \text{otherwise and provided } f'(t) \text{ exists.} \end{cases}$$

Define $f_\epsilon(x) := f(0) + \int_0^x \phi_\epsilon(t)dt$. Then $f_\epsilon \in Lip_1$ and

$$|f(x) - f_\epsilon(x)| = |\int_0^x f'(t) - \phi_\epsilon(t)dt| \le \int_0^1 |f'(t) - \phi_\epsilon(t)|dt = 4\epsilon.$$

On I, for every $r \in [-1 + 1/n, 1 - 1/n]$, the function $f_\epsilon(x) - rx$ is not nondecreasing on I because on $(x_0 - \epsilon, x_0)$ it has derivative $-1 - r \le -1/n$. Thus A_I^n is nowhere dense and $A_I = \bigcup_{n=2}^{\infty} A_I^n$ is of first category. Now let $\{I_k\}$ be an enumeration of those open subintervals of $[0,1]$ having rational endpoints. Set $A := \bigcup_{k=1}^{\infty} A_{I_k}$. Then A is a first category set in Lip_1. Similarly, we show that

$$B := \{f \in Lip_1 : f(x) - rx \text{ is nonincreasing on some open subinterval of } [0,1] \text{ for some } r \in (-1,1)\},$$

is of first category in Lip_1. If $f \in Lip_1 \setminus (A \cup B)$, then for every $r \in (-1,1)$ the function $f(x) - r \cdot x$ is nowhere monotone on $[0,1]$.

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